Стационарлық емес ағыс жағдайында жүйенің микробөлшектерін қозғалысының (ретсіз, реттелген, абсолюттік) әртүрле түрлерге арналған энергияның теңдеулерін қорыту беріледі. Сейілтенің центрлеген толқынында стационарлық емес ағын параметрлері үлгісін есептеу алынған теңдеулерді қолданғанда байқалады.

The inference of the energy equations for various kinds of movement of system microparticles (for chaotic, directional, absolute) in case of non-stationary current is given. Use of the received equations is shown on the example of calculation of a non-stationary stream parameters in the centered rarefaction wave.

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V. V. Ryndin

INTEGRATION OF EQUATION OF MOTION OF VISCOUS COMPRESSIBLE FLUID (NAVIER-STOKES) LENGTHWAYS TRAJECTORIES AND STREAMLINES

Method is founded on multiplication of members of the vectorial equation of Navier-Stokes on an elementary transition $d\vec{r}$ *of an element of a fluid, or on an elementary segment* $\delta\vec{r}$ *of a streamline; thus the introduction of potential functions of forces of pressure, mass forces and velocity is not required. The method of calculation of the integrals, which are going into obtained equations, on an example of the centered wave of pressure is given.*

In existing educational courses of a mechanics of a fluid and gas [1, 2, 3] the integration of equations of motion of Euler and Navier-Stokes is yielded only along a streamline. A method of an integration of equations of motion of Navier-Stokes along a trajectory, and also new method of an integration along a streamline is explained below.

Let's consider unsteady motion of viscous compressible fluid, for which one the equation of motion of Navier-Stokes can be written to an aspect of the vectorial equation expressing balance of specific forces (N/kg), operating on a device (macroscopic particle) of a fluid of a unit mass, –

$$\vec{f}_{\text{grav}} - \frac{1}{\rho}\text{grad}p + н[\nabla^2\vec{c} + \frac{1}{3}\nabla(\nabla\vec{c})] = \frac{d\vec{c}}{dt}$$

or

$$\vec{f}_{\text{grav}} + \vec{f}_{\text{p}} + \vec{f}_{\text{fr}} + \vec{f}_{\text{iner}} = 0, \qquad (1)$$

where $\vec{f}_{\text{grav}} = \vec{F}_{\text{grav}}/m$ – specific gravity force;

$\vec{f}_{\text{p}} = \vec{F}_{\text{p}}/m = -\frac{1}{\rho}\text{grad}p = -\upsilon\,\text{grad}p$ – specific resultant of forces of pressure (hydrodynamic of pressure); $x = V/m$ – specific volume, m^3/kg;

$\vec{f}_{\text{fr}} = \vec{F}_{\text{fr}}/m = \nu[\nabla^2\vec{c} + \frac{1}{3}\nabla(\nabla\vec{c})]$ – specific force of viscous friction;

$\vec{f}_{\text{iner}} = \vec{F}_{\text{iner}}/m = -d\vec{c}/dt$ – specific force of inertia.

Integration of the differential equation (1) in the beginning we shall conduct **along a trajectory** of a macroscopic particle, that it is much easier, than integration along a streamline, and it has concrete physical sense. For this purpose we shall multiply scalarly all member of equations (1) on partial transition of a fluid particle $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$:

$$\vec{f}_{\text{grav}}d\vec{r} + \vec{f}_{\text{p}}d\vec{r} + \vec{f}_{\text{fr}}d\vec{r} + \vec{f}_{\text{iner}}d\vec{r} = 0. \qquad (2)$$

From the physical point of view such multiplication means passage from specific forces to specific works (J/kg), which one are committed by these forces at transition of a element of a fluid of a unit mass on distance $d\vec{r}$. Let's consider these works.

If in quality of spatial (volumetric) forces takes only force of the gravity, directional vertically downwards $\vec{f}_{\text{grav}} = \vec{g}m/m = -\vec{k}g$,

where $\vec{g} = -\vec{k}g$ – the free fall acceleration, then specific work of gravity will be defined so:

$$\vec{f}_{\text{grav}}d\vec{r} = -\vec{k}g(dx\vec{i} + dy\vec{j} + dz\vec{k}) = -g\,dz.$$

The specific work of resultant of forces of pressure operating on the verges of a macroscopic particle of a fluid, will be defined so:

$$\vec{f}_{\text{p}}d\vec{r} = -\upsilon\,\text{grad}p\,d\vec{r} = -\upsilon(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz).$$

The specific work of viscous forces (friction work)

$$\vec{f}_{\text{fr}}d\vec{r} = \nu[\nabla^2\vec{c} + \frac{1}{3}\nabla(\nabla\vec{c})]\,d\vec{r} = \delta w'_{\text{fr}} = -\delta w_{\text{fr}}.$$

(here w' – work of external forces, $w = -w'$ – work of interior forces).

Specific work of an inertial force with the count that $\mathrm{d}\vec{r}/\mathrm{d}t = \vec{c}$, will be defined so:

$$\vec{f}_{\text{iner}}\mathrm{d}\vec{r} = -(\mathrm{d}\vec{c}/\mathrm{d}t)\,\mathrm{d}\vec{r} = -\vec{c}\,\mathrm{d}\vec{c} = -\mathrm{d}c^2/2$$

(as we see, she is equal of decrease of a specific kinetic energy of an element of a fluid).

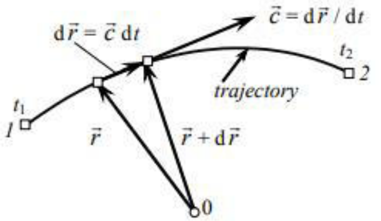
Substituting the retrieved expressions for works in the equation (2), we shall receive

$$-g\mathrm{d}z - \upsilon\left(\frac{\partial p}{\partial x}\mathrm{d}x + \frac{\partial p}{\partial y}\mathrm{d}y + \frac{\partial p}{\partial z}\mathrm{d}z\right) - \delta w_{\text{fr}} = \mathrm{d}c^2/2. \qquad (3)$$

Equation (3) expresses law of a modification of a kinetic energy of a element of a fluid of an unit mass during dt.

Integrating equation (3) along trajectory from point *1* up to point *2* (**figure 1**), we shall receive final values of works and modification of a kinetic energy of an element of a fluid of a unit mass during Δt:

$$g(z_2 - z_1) + (c_2^2 - c_1^2)/2 + w_{\text{fr}} = -\int_1^2 \upsilon\left(\frac{\partial p}{\partial x}\mathrm{d}x + \frac{\partial p}{\partial y}\mathrm{d}y + \frac{\partial p}{\partial z}\mathrm{d}z\right). \qquad (4)$$



1 – position of a macroscopic particle in moment of time t_1;
2 – position of a macroscopic particle in moment of time t_2;
$\mathrm{d}\vec{r}$ – elementary transition of a macroscopic particle during $\mathrm{d}t$.

Figure 1 – To an integration along a trajectory

If to take a relation for a differential of pressure

$$\mathrm{d}p = \frac{\partial p}{\partial t}\mathrm{d}t + \frac{\partial p}{\partial x}\mathrm{d}x + \frac{\partial p}{\partial y}\mathrm{d}y + \frac{\partial p}{\partial z}\mathrm{d}z,$$

then the equation (4) is possible to give and such view:

$$g(z_2 - z_1) + (c_2^2 - c_1^2)/2 + w_{\text{fr}} + \int_1^2 \upsilon\,\mathrm{d}p - \int_1^2 \upsilon\frac{\partial p}{\partial t}\mathrm{d}t = 0. \qquad (5)$$

If a fluid is incompressible (υ = const and inviscid $\nabla\vec{c} = \mathrm{div}\vec{c} = 0$), then

$$\int_1^2 x\,\mathrm{d}p = (p_2 - p_1)x = (p_2 - p_1)/\rho$$

and the equation (5) will accept such view:

$$z_1 + \frac{p_1}{\rho g} + \frac{c_1^2}{2g} = z_2 + \frac{p_2}{\rho g} + \frac{c_2^2}{2g} + h_{\text{iner}} + h_{\text{fr}}, \qquad (6)$$

where quantity $h_{\text{iner}} = -\frac{1}{\rho g}\int_1^2 \frac{\partial p}{\partial t}\mathrm{d}t$ it is possible to term as an «inertial head»;

$$h_{\text{fr}} = \frac{w_{\text{fr}}}{g} = -\frac{\nu}{g}\int_1^2 \nabla^2\vec{c}\,\mathrm{d}\vec{r} = -\frac{\nu}{g}\int_1^2\left(\frac{\partial^2 c_x}{\partial x^2}\mathrm{d}x + \frac{\partial^2 c_y}{\partial y^2}\mathrm{d}y + \frac{\partial^2 c_z}{\partial z^2}\mathrm{d}z\right) -$$

specific weight work of frictional forces acting on a element of a fluid of a single weight, J/N (friction head).

The equation (6) it is possible to term as a Bernoulli's relation for an elementary trickle in case of non-stationary fluxion of viscous incompressible fluid. In case of steady flow hiner = 0 and the equation (6) becomes of a known Bernoulli's relation for a elementary trickle of viscous incompressible fluid:

$$z_1 + \frac{p_1}{\rho g} + \frac{c_1^2}{2g} = z_2 + \frac{p_2}{\rho g} + \frac{c_2^2}{2g} + h_{\text{fr}}. \qquad (7)$$

For an **integration** of the equation (1) **along a streamline** we shall multiply scalarly all its members on a elementary segment of a streamline $\delta\vec{r} = \delta x\vec{i} + \delta y\vec{j} + \delta z\vec{k}$ – elementary segment, is conducting in space in the given instant (he the transition of a separate fluid particle does not characterize, as joins two different fluid particles in the given instant), –

$$\vec{f}_{\text{grav}}\delta\vec{r} + \vec{f}_{\text{p}}\delta\vec{r} + \vec{f}_{\text{fr}}\delta\vec{r} + \vec{f}_{\text{iner}}\delta\vec{r} = 0. \qquad (8)$$

Multiplying specific forces on segments of a streamline[1], we obtain works, which one could be accomplished, if the fluid particle instantaneously moved along a streamline under an operation of the forces, operating lengthways her in the given instant, that is, these works are only calculated, or conditional, as the real works are gained at multiplication of forces to real transition of a particle $d\vec r$, committed by her for some time interval.

In case of stationary (steady-state) flow, streamlines and allocation of values of flow parameters along streamlines coincide with trajectories and allocation of values of the corresponding parameters along trajectories. Only in this case, conditional works, committed along streamlines gain sense of real works calling a modification of a kinetic energy of a fluid particle at her moving along a trajectory.

In conditionality of works committed along a streamline, concluded difficulty of energy interpretation of members which are included in the integral Bernoulli, obtained at an integration of the equations of Euler along a streamline. These shortages the method of an integration of equations of motion along trajectories surveyed above is dispossessed, that allows to consider his as basic for deriving a Bernoulli's relation (7) of the equation (1).

Let's uncover expressions for conditional works which are included in the equation (8).

Conditional specific work of gravity

$$\vec f_{\text{grav}}\delta\vec r = \vec g\delta\vec r - \vec k g(\delta x\vec i + \delta y\vec j + \delta z\vec k) = -g\,\delta z \,.$$

The conditional specific work of resultant of forces of pressure

$$\vec f_{\text{p}}\delta\vec r = -\upsilon\,\text{grad}p\,\delta\vec r = -\upsilon\left(\frac{\partial p}{\partial x}\delta x + \frac{\partial p}{\partial y}\delta y + \frac{\partial p}{\partial z}\delta z\right) = -\frac{1}{\rho}\text{d}p\,,$$

as the change of pressure along a streamline happens only at the expense of icrease of coordinates, and the change of pressure from time to be equal zero ($\frac{\partial p}{\partial t}\text{d}t = 0$).

The conditional specific work of viscous forces (conditional friction work)

$$\vec f_{\text{fr}}\delta\vec r = \text{н}[\nabla^2\vec c + \frac{1}{3}\nabla(\nabla\vec c)]\delta\vec r = -\delta w'_{\text{fr}}\,.$$

1 Streamline – line, in each point by which one the vector of velocity in the given instant is tangential to her. Therefore, the streamline is formed by an whole of particles taken in one same instant, the velocities which one are tangential to this line. The trajectory – line, which is described one the same particle at the her moving for some time interval.

(in case of non-stationary fluxion the work of frictional force along a trajectory is not equal to conditional work of frictional force along a streamline $\delta w_{\text{fr}} \neq \delta w'_{\text{fr}}$, because the forces of the viscosities operating along a streamline between sections *1-2* in the given instant, are not equal to forces of viscosity operating on a fluid particle at her moving between sections *1-2* during dt).

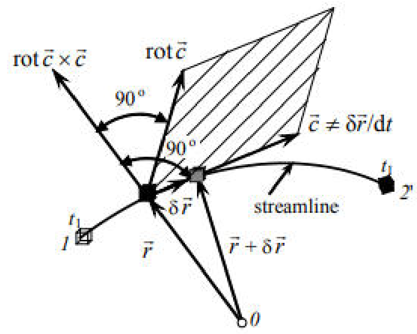
As $\delta\vec r/\text{d}t \neq \vec c$, then and $\vec f_{\text{iner}}\delta\vec r = -(\text{d}\vec c/\text{d}t)\delta\vec r \neq -\text{d}c^2/2$, therefore we shall take advantage of transformations Gromeco-Lamba for acceleration of a particle

$$\text{d}\vec c/\text{d}t = \partial\vec c/\partial t + \text{grad}(c^2/2) + \text{rot}\,\vec c\times\vec c\,.$$

Then

$$(\text{d}\vec c/\text{d}t)\delta\vec r = (\partial\vec c/\partial t)\delta\vec r + \text{grad}(c^2/2)\delta\vec r + (\text{rot}\,\vec c\times\vec c\,)\delta\vec r = (\partial\vec c/\partial t)\delta\vec r + \text{d}c^2/2\,,$$

as by virtue of made above notes $\text{grad}(c^2/2)\delta\vec r = \text{d}c^2/2$ and by virtue of a pependicularity of vectors ($\text{rot}\,\vec c\times\vec c$) and $\delta\vec r$ (**figure 2**) the dot product of these vectors is equal to zero $(\text{rot}\,\vec c\times\vec c\,)\delta\vec r = 0$.



1 – position of the first macroscopic particle in an instant t_1;
2' – position of the second macroscopic particle in an instant t_1;
$\delta\vec r$ – elementary distance between two macroscopic particles in an instant t_1.

Figure 2 – To an integration along a streamline

With the count of made notes the equation (8) will accept an aspect

$$-g\mathrm{d}z-\frac{1}{\rho}\mathrm{d}p-\delta w'_{\text{fr}}=(\partial\vec{c}/\partial t)\delta\vec{r}+\mathrm{d}c^2/2.$$

Integrating this equation from a point *1* up to a point *2'* (the point *2'* on a streamline has the same coordinates, as point *2* on trajectory of a particle, but as the instants are various, also state variables of a particle in points *2* and *2'* will be different, excluding of height z) on a streamline, we shall receive

$$g(z_{2'}-z_1)+\int_1^{2'}\frac{\mathrm{d}p}{\rho}+(c_{2'}^2-c_1^2)/2+w'_{\text{fr}}+\int_1^{2'}\frac{\partial\vec{c}}{\partial t}\delta\vec{r}=0 \tag{9}$$

At $\rho=\text{const}$ the equation (9) will accept an aspect of a Bernoulli's relation for a elementary tube of a current in case of non-stationary fluxion of viscous incompressible fluid [1, 3]

$$z_1+\frac{p_1}{\rho g}+\frac{c_1^2}{2g}=z_2+\frac{p_{2'}}{\rho g}+\frac{c_{2'}^2}{2g}+h'_{\text{iner}}+h'_{\text{fr}}, \tag{10}$$

where $h'_{\text{iner}}=\int_1^{2'}\frac{\partial\vec{c}}{\partial t}\delta\vec{r}$ – so-called inertial head;

$h'_{\text{fr}}=\frac{w'_{\text{fr}}}{g}=-\frac{\nu}{g}\int_1^{2'}\nabla^2\vec{c}\,\delta\vec{r}$ – conditional friction heads.

In case of steady flow the equation (10), obtained at an integration along a streamline, coincides with the equation (7), obtained at an integration along a trajectory.

Singularities of calculation of integrals included in equations (4), (5) and (9), we shall show on an example of calculation of parameters of the unsteady flow, originating in a centered wave of pressure.

The centered wave of pressure can be received in a tube in front of the cylinder piston at his moving under the particular law, when all characteristics (straight lines along which one all parameters of gas remain stationary values) intersect in one point $A(x_0,t_0)$ (**figure 3**). Coordinate of a point A can be spotted under the formula $x_0=a_\text{n}t_0$, where a_n – speed of sound in nonperturbed gas, when the flow velocity equal zero $a_\text{n}=0$.

As the one-dimensional fluxion of gas in a wave of pressure is considered isentropical (without exterior heat exchange and without friction) up to the moment of formation of a shock wave in a point A, then the equation of a modification of a kinetic energy of a macroscopic particle (4) at forgetfulness by work of gravity will accept such aspect:

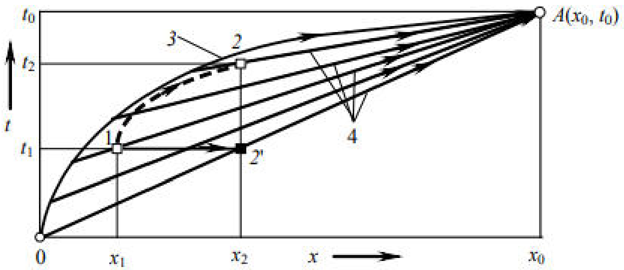
$$c_2^2/2-c_1^2/2=-\int_1^2\upsilon\frac{\partial p}{\partial x}\mathrm{d}x, \tag{11}$$

and the equation (5) with the count that connection between a denseness (specific volume) and pressure is erected by the equation of an isentrope $p/\rho^k=\text{const}$ and

$$\int_1^2 x\,\mathrm{d}p=\int_1^2\mathrm{d}p/\rho=\frac{k}{k-1}\frac{p_2}{\rho_2}-\frac{k}{k-1}\frac{p_1}{\rho_1}=\frac{a_2^2}{k-1}-\frac{a_1^2}{k-1}$$

(where $k=c_p/c_\upsilon$; $a=\sqrt{kp/\rho}$ – the speed of sound), will accept such aspect:

$$\frac{a_1^2}{k-1}+\frac{c_1^2}{2}=\frac{a_2^2}{k-1}+\frac{c_2^2}{2}-\int_1^2 x\frac{\partial p}{\partial t}\mathrm{d}t. \tag{12}$$



1-2 – trajectory of a macroscopic particle;
1-2' – a streamline; *3* - path of the cylinder piston;
4 – characteristics

Figure 3 – To calculation of parameters of the unsteadyflow originating in a centered wave of pressure

The equation (9), defining relation of parameters of gas on a streamline (line *1-2'* in figure 3), with the count that in a point *2'* gas is in a nonperturbed state ($c_{2'}=c_\text{n}=0,\ a_{2'}=a_\text{n}$), will accept such aspect:

$$\frac{a_1^2}{k-1}+\frac{c_1^2}{2}=\frac{a_\text{n}^2}{k-1}++\int_{x_1}^{x_2}\left(\frac{\partial c}{\partial t}\right)_{t_1}\mathrm{d}t. \tag{13}$$

Let's consider a method of calculation of an integral which is included in the equation (11). Using association of pressure in the centered compression wave from coordinates and time

$$p=(5/6)^7 p_n\left(1+\frac{1}{5a_n}\frac{x-x_0}{t-t_0}\right)^7$$

(all associations for a wave of pressure are taken from work [4, 5] for $k=1,4$), we find:

$$\frac{\partial p}{\partial x}=(7/5)(5/6)^6\frac{p_n}{a_n}\left(1+\frac{1}{5a_n}\frac{x-x_0}{t-t_0}\right)^6/(t-t_0), \qquad (14)$$

and with the count of the equation of a trajectory for a macroscopic particle of gas

$$x=x_0-\left[6a_1\left(\frac{t_0-t_1}{t_0-t}\right)^{\frac{1}{6}}-5a_n\right](t_0-t) \qquad (15)$$

we transfer to one variable in the equation (14)

$$\frac{\partial p}{\partial x}=(7/5)(5/6)^7\frac{p_n}{a_n}\left(\frac{6a_1}{5a_n}\right)^6(t_1-t_0)/(t-t_0)^2.$$

Specific volume of gas is defined from the equation of an isentrope with the count of association of pressure in a macroscopic particle from time from a beginning of moving it

$$\frac{p}{p_n}=\left(\frac{a_1}{a_n}\right)^7\left(\frac{t_1-t_0}{t-t_0}\right)^{\frac{7}{6}}=\left(\frac{a}{a_n}\right)^7$$

and of relation for a speed of sound in nonperturbed gas $a_n^2=kp_n\upsilon_n$

$$\upsilon=\upsilon_n\left(\frac{p_n}{p}\right)^{\frac{1}{k}}=\frac{5a_n}{7p_n}\left(\frac{a_1}{a_n}\right)^{-5}\left(\frac{t_1-t_0}{t-t_0}\right)^{-\frac{5}{6}}.$$

Differentiating the equation (15), we define association of elementary transition of a macroscopic particle from time

$$dx=5a_n\left[\frac{a_1}{a_n}\left(\frac{t_1-t_0}{t-t_0}\right)^{\frac{1}{6}}-1\right]dt.$$

Substituting the corresponding expressions in the equation (11) and integrating, we shall receive:

$$c_2^2/2-c_1^2/2=-\int_1^2\upsilon\frac{\partial p}{\partial x}dx=(25/2)\left\{a_1^2\left[\left(\frac{t_1-t_0}{t_2-t_0}\right)^{\frac{1}{3}}-1\right]-2a_na_1\left[\left(\frac{t_1-t_0}{t_2-t_0}\right)^{\frac{1}{6}}-1\right]\right\}=$$

$$=\frac{2}{(k-1)^2}[(a_2-a_n)^2-(a_1-a_n)^2]. \qquad (16)$$

On the other hand, using the equation of a relation of parameters on characteristics

$$c_1=\frac{2}{k-1}(a_1-a_n)\text{ and }c_2=\frac{2}{k-1}(a_2-a_n), \qquad (17)$$

find

$$c_2^2/2-c_1^2/2=\frac{2}{(k-1)^2}[(a_2-a_n)^2-(a_1-a_n)^2],$$

that coincides with expression (16). Therefore, the integration along a trajectory of a particle in a wave of pressure is performed correctly.

Let's consider a method of calculation of an integral which is going into the equation (13). Using association of a velocity of a macroscopic particle on time and coordinates

$$c=\frac{5}{6}\left(\frac{x-x_0}{t-t_0}-a_n\right),$$

find

$$\frac{\partial c}{\partial t}=\frac{5}{6}\frac{x-x_0}{(t-t_0)^2}$$

and

$$\int_{x_1}^{x_2}\left(\frac{\partial c}{\partial t}\right)_{t_1}dx=\frac{5}{12}\left[\left(\frac{x_1-x_0}{t_1-t_0}\right)^2-\left(\frac{x_2-x_0}{t_2-t_0}\right)^2\right]=\frac{5}{12}[(c_1+a_1)^2-a_n^2].$$

Putting a value of this integral into the equation (13) and solving him concerning a velocity c1, we shall receive

$$c_1=5(a_1-a_n)=\frac{2}{k-1}(a_1-a_n),$$

that coincides with the equation of a relation of parameters on characteristic (17).

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Ағын сызығын және траекторияны жағалай тұтқыр сығатын сұйықтың (Навье-Стокса) қозғалысының теңдеуін интегралдау

С. Торайғыров атындағы Павлодар мемлекеттік университеті, Павлодар қ.
Материал 07.01.13 редакцияға түсті.

В. В. Рындин

Интегрирование уравнения движения вязкой сжимаемой жидкости (Навье-Стокса) вдоль траектории и линии тока

Павлодарский государственный университет имени С. Торайгырова, г. Павлодар.
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Ағының сызығы элементар кесіндісі әлде элементар ауыстырушылықпен Навье-Стокса теңдеудің мүшелерін көбейтумен әдіс негіздеді. Сонымен қатар жылдамдық және қысым күштерінің потенциалды функциялар кіріспені керек болмайды. Қысымның центрлеген толқынын үлгісінде интегралдар есептеудің әдістерін беріледі.

Интегрирование вдоль траектории основано на умножении членов уравнения Навье-Стокса на элементарное перемещение , а вдоль линии тока – на элементарный отрезок линии тока. При этом не требуется введения потенциальных функций сил давления и скорости. Даётся методика вычисления интегралов на примере центрированной волны давления.