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PROPAGATION OF ELECTROMAGNETIC WAVES IN CHOLESTERIC LIQUID CRYSTALS

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Fundamental properties of solutions of Maxwell's equations describing the propagation of electromagnetic waves in a cholesteric liquid crystal with tensor characteristics depending on one of the spatial coordinates (the Z-axis is chosen) are investigated. The matrix of the coefficients, the structure of the matrix of Maxwell's equations, and the dispersion equations for an anisotropic liquid cholesteric medium are obtained.

Keywords: anisotropic medium, Maxwell's equations, electromagnetic waves, liquid crystals, cholesterics, dispersion equations, periodic structure, matricant.

INTRODUCTION

Cholesteric liquid crystals (CLC) are anisotropic inhomogeneous materials with interesting and useful properties in the visible range of the electromagnetic spectrum. Christou *et al*. [1] used results of analysis of eigenmodes in a thin homogeneous intermediate layer of an CLC cell (liquid crystal consisted of several intermediate layers) to obtain expressions for the guided field. The cell was clamped between dielectric layers and was excited by an elliptically polarized obliquely incident plane wave. The solution of the linear system of equations gave the reflection and transmission coefficients in two main planes and the expansion coefficients for modal fields inside the CLC and dielectric layers. The propagation of plane electromagnetic waves through a liquid crystal layer was studied in [2]; special attention was given to the problem of optimization of the transmitted radiation intensity. The anisotropy of the liquid crystal layer controlled either by fastening to guiding glass plates located between the layers or by applying an external electromagnetic field allowed the layer orientation to be adjusted to maximize or minimize the intensity of radiation at a preset wavelength transmitted through the layer.

The nonlinear spin dynamics of the Heisenberg helimagnet was studied during electromagnetic wave propagation in [3]. The basic dynamic equation of spin evolution governed by the Landau–Lifshits equation was analogous to the directive twist dynamics in cholesteric liquid crystals. It was revealed that both the magnetization, and the magnetic field itself were modulated as soliton modes in the process of electromagnetic wave propagation through the medium because of amplitude fluctuations introduced in the tail of the field. The quasi-isotropic approach (QIA) of geometrical optics which describes the properties of electromagnetic waves in weakly anisotropic media, including weakly isotropic fibers, liquid crystals, and weakly magnetized plasma, was stated in [4]. The QIA equations followed directly from Maxwell's equations and had the form of the first-order coupled equations for the transverse electromagnetic field components. The QIA applied to the magnetized plasma describes the joint action of the Faraday and Cottoun–Mouton phenomena and provides the theoretical basis for plasma polarimetry in the far field infra-red (FIR) and microwave zones. Aksenova *et al*. [5, 6] considered the Green's function and the waveguide propagation of the electromagnetic field in cholesteric liquid crystals with step larger than the wavelength. This function was constructed using the solution of Maxwell's equations. They also analyzed in detail its behavior in the far field zone.

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A periodic system differs from an anisotropic medium by the presence of discontinuity on the wave vector surface and kink of the beam vector surface. The phenomena of plane wave reflection from and transmission through plates made of artificial bianisotropic homogeneous media were theoretically investigated in [7]. The case of the isotropic basic medium possessing only dielectric properties was studied. The feasibility of creation of the twirled omega structure with microwave properties analogous to those of the optical properties of cholesteric liquid crystals was demonstrated. Analytical solutions for the propagation of electromagnetic waves parallel to the spiral axis in a continuously twirled biaxial dielectric medium as well as along the screw axis of a supercholesteric material were presented in [8, 9]. Two optical axes of this unidirectional inhomogeneous medium lay in the plane perpendicular to the spiral axis and had a constant phase shift. The 4×4 optics matrix method was used. Exact solutions of the problem of axial wave propagation were obtained.

Guided electromagnetic waves (GEW) at optical frequencies propagating in cholesteric liquid crystals of finite thickness were theoretically investigated in [10] under conditions of total internal reflection from both boundaries. Two types of the interface were discussed: CLC-metal and CLC-dielectric. It was shown that waves of two types could propagate in such films, namely, attenuated and non-attenuated modes. The second-order diffraction of the optical surface guided electromagnetic waves (SGEW) in cholesterics was studied in [11] in the context of dynamic diffraction theory. The dispersion equations were derived and analyzed. The frequency bands and the propagation directions (relative to the orientation of the director on the cholesteric surface) of the SGEW were determined. The structure and polarization of surface electromagnetic waves (SEV) propagating along the interface between the CLC and the substrate with a small refractive index were studied in [12]. It was shown that these SEV are effectively generated by the attenuated total reflection method. Processes in elastic anisotropic and anisotropic dielectric media and propagation of waves in anisotropic plates, electromagnetic waves in media with magnetoelectric effect [14–17], and waves in liquid crystals and thermoelastic media [18–20] were considered in [13] using the matricant method. Results of uniform description of wave processes in media with different physicomechanical properties were reported at the XVth All-Russian School-Seminar “Physics and Application of Microwaves” named after A. P. Sukhorukov (Waves-2016) [21].

PURPOSE, PROBLEMS, AND NOVELTY OF THE RESEARCH

The purpose of the present work is a study of electromagnetic wave propagation in cholesteric liquid crystals based on the analytical matricant method. To achieve this purpose, the following problems have been solved: the system of the first order differential equations was constructed for cholesteric liquid crystals based on Maxwell’s equations using the method of separation of variables. This system described the propagation of electromagnetic waves in cholesteric liquid crystals. From this system of equations, the matrix of the coefficients was derived whose elements provided the basis for further research. Then the matricant (normalized solution of the system of differential equations) was obtained.

The scientific novelty of the present work consists in the application of liquid crystals in electronic instrument making which uses the special features and laws of electromagnetic wave propagation. Among the advantages of application of the matricant method are introduction of the notion of the matricant structure and its definition has allowed us to generalize the classical methods developed by Brillouin and Parodi for discrete periodic structures to periodic inhomogeneous media; the possibility to halve the degree of the characteristic equation describing wave dispersion; the possibility of method application to a study of the electromagnetic and elastic (mechanical) wave propagation in different media, including elastic, thermoelastic, piezoelastic, piezoelectric, electroelastic, piezomagnetic, magnetoelectric, thermopiezoelectric, and liquid crystals.

METHOD OF RESEARCH

As a method of research, we used the matricant method that allowed us to obtain exact analytical solutions of differential equations describing coupled processes in media with piezoelectric, piezomagnetic, and thermopiezoelectric properties to solve problems formulated in the project.

Basic equations and relationships. Matrix of the coefficients

The phase of the substance the molecules of which are stretched along a certain direction and are between the liquid and solid states is called liquid crystal one. If at the known coordinate *z* the orientation of molecules is observed, it will change by a spiral attendant to changes of *z*. Such substances refer to cholesteric liquid crystals (or cholesterics). The unit vector defining orientation of molecules is called the director. Its direction in the cholesteric is determined by the following formulas [22]:

$$n_x = n_1 = \cos\left(q_0 z + \varphi\right),$$
$$n_y = n_2 = \sin\left(q_0 z + \varphi\right), \tag{1}$$
$$n_z = n_3 = 0.$$

The spatial period of the director spiral is about $3 \cdot 10^3$ Å, that is, greater than the atom size. The dielectric permittivity tensor depends only on the components of the vector [22]:

$$\varepsilon_{ij} = \left(\delta_{ij} - n_i n_j\right)\varepsilon_{\perp} + n_i n_j \varepsilon_{\mathrm{II}}, \tag{2}$$

where $\varepsilon_{\mathrm{II}}$ and $\varepsilon_{\perp}$ are the relative dielectric permittivity components along the *Z* axis and perpendicular to it. Let us write the dielectric permittivity tensor components using formula (2):

$$\varepsilon_{11} = \left(1 - n_1^2\right)\varepsilon_{\perp} + n_1^2 \varepsilon_{\mathrm{II}} = \varepsilon_{\perp} + \left(\varepsilon_{\mathrm{II}} - \varepsilon_{\perp}\right) n_1^2, \quad \varepsilon_{12} = -n_1 n_2 \varepsilon_{\perp} + n_1 n_2 \varepsilon_{\mathrm{II}} = \left(\varepsilon_{\mathrm{II}} - \varepsilon_{\perp}\right) n_1 n_2,$$
$$\varepsilon_{13} = 0, \quad \varepsilon_{21} = \varepsilon_{12}, \quad \varepsilon_{22} = \left(1 - n_2^2\right)\varepsilon_{\perp} + n_2^2 \varepsilon_{\mathrm{II}} = \varepsilon_{\perp} + \left(\varepsilon_{\mathrm{II}} - \varepsilon_{\perp}\right) n_2^2, \tag{3}$$
$$\varepsilon_{23} = 0, \quad \varepsilon_{31} = 0, \quad \varepsilon_{32} = 0, \quad \varepsilon_{33} = \varepsilon_{\perp}.$$

The structure of the dielectric permittivity tensor has the following form:

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}. \tag{4}$$

Considering formula (1), from formulas (3) we obtain

$$\varepsilon_{11} = \varepsilon_{\perp} + \left(\varepsilon_{\mathrm{II}} - \varepsilon_{\perp}\right)\cos^2\left(q_0 + \varphi\right),$$
$$\varepsilon_{12} = \left(\varepsilon_{\mathrm{II}} - \varepsilon_{\perp}\right)\cos\left(q_0 + \varphi\right)\sin\left(q_0 + \varphi\right),$$
$$\varepsilon_{22} = \varepsilon_{\perp} + \left(\varepsilon_{\mathrm{II}} - \varepsilon_{\perp}\right)\sin^2\left(q_0 + \varphi\right),$$
$$\varepsilon_{33} = \varepsilon_{\perp}.$$

The form of the dielectric permittivity tensor corresponds to that of the anisotropic dielectric medium of a monoclinic system. In this case, the second order symmetry axis is parallel to the Z axis.

Let the values of the dielectric permittivity and magnetic permeability be known. The properties of the medium are given, even if one of them is a scalar. Based on this, we can write material equations for the dielectric anisotropic medium in the form

$$D_i = \varepsilon_0 \varepsilon_{ij} E_j\,,\ \varepsilon_{ij} = \left(\omega, \boldsymbol{r}\right),\ B_i = \mu_0 \mu_{ij} H_j\,,\ \mu_{ij} = \left(\omega, \boldsymbol{r}\right). \tag{5}$$

Let us assume that these tensors depend only on one spatial coordinate Z, that is, the medium is inhomogeneous along the Z axis. If the volume charge density ρ and the current density j are equal to zero, the system of Maxwell's equations takes the form

$$\operatorname{rot} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\,, \tag{6}$$

$$\operatorname{div} \boldsymbol{B} = 0\,, \tag{7}$$

$$\operatorname{rot} \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}\,, \tag{8}$$

$$\operatorname{div} \boldsymbol{D} = 0\,, \tag{9}$$

where $\boldsymbol{E}$ is the electric field strength, $\boldsymbol{B}$ is the magnetic field induction vector, $\boldsymbol{H}$ is the magnetic field strength, and $\boldsymbol{D}$ is the electric field induction vector.

Considering the foregoing, the wave fields $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{B}$, and $\boldsymbol{D}$ can be represented in harmonic form as

$$\boldsymbol{F} = \boldsymbol{F}(\omega, z) e^{i\omega t \pm i k_x x \pm i k_y y}\,, \tag{10}$$

where ω is the frequency and k_x and k_y are the wave vector components along the X and Y axes, respectively. The properties of the medium do not depend on x and y coordinates. We can write the system of the first order equations describing the propagation of electromagnetic waves in the cholesteric crystal in the form

$$\begin{aligned}
\frac{dE_y}{dZ} &= i\left(\omega\mu_0\mu - \frac{k_y^2}{\omega\varepsilon_0\varepsilon_z}\right) H_x + i\frac{k_x k_y}{\omega\varepsilon_0\varepsilon_z} H_y,\\
\frac{dH_x}{dZ} &= i\left(\omega\varepsilon_0\varepsilon_y - \frac{k_x^2}{\omega\mu_0\mu}\right) E_y + i\left(\omega\varepsilon_0\varepsilon_{yx} + \frac{k_x k_y}{\omega\mu_0\mu}\right) E_x,\\
\frac{dH_y}{dZ} &= -i\left(\omega\varepsilon_0\varepsilon_{xy} + \frac{k_x k_y}{\omega\mu_0\mu}\right) E_y - i\left(\omega\varepsilon_0\varepsilon_x - \frac{k_x^2}{\omega\mu_0\mu}\right) E_x,\\
\frac{dE_x}{dZ} &= -i\frac{k_x k_y}{\omega\varepsilon_0\varepsilon_z} H_x - i\left(\omega\mu_0\mu - \frac{k_x^2}{\omega\varepsilon_0\varepsilon_z}\right) H_y.
\end{aligned} \tag{11}$$

or in the matrix form

$$\frac{d}{dZ}\begin{pmatrix} E_y \\ H_x \\ H_y \\ E_x \end{pmatrix} = \begin{pmatrix} 0 & b_{12} & b_{13} & 0 \\ b_{21} & 0 & 0 & b_{24} \\ -b_{24} & 0 & 0 & b_{34} \\ 0 & -b_{13} & b_{43} & 0 \end{pmatrix}\begin{pmatrix} E_y \\ H_x \\ H_y \\ E_x \end{pmatrix}. \tag{12}$$

Let us write Eq. (12) as

$$\frac{d\boldsymbol{u}_x}{dz} = \hat{B}\boldsymbol{u} \ ,$$

$$\boldsymbol{u} = \left(E_y, H_x, H_y, E_x\right)^t , \tag{13}$$

where the matrix $\hat{B}$ has the structure

$$\hat{B} = \begin{pmatrix} 0 & b_{12} & b_{13} & 0 \\ b_{21} & 0 & 0 & b_{24} \\ -b_{24} & 0 & 0 & b_{34} \\ 0 & -b_{13} & b_{43} & 0 \end{pmatrix}, \tag{14}$$

in which the coefficients of matrix (14) have the form

$$b_{12} = i\left(\omega\mu_0\mu - \frac{k_y^2}{\omega\varepsilon_0\varepsilon_z}\right), \quad b_{13} = i\frac{k_x k_y}{\omega\varepsilon_0\varepsilon_z}, \quad b_{21} = i\left(\omega\varepsilon_0\varepsilon_y - \frac{k_x^2}{\omega\mu_0\mu}\right),$$

$$b_{24} = i\left(\omega\varepsilon_0\varepsilon_{yx} + \frac{k_x k_y}{\omega\mu_0\mu}\right), \quad b_{31} = -b_{24}, \quad b_{34} = -i\left(\omega\varepsilon_0\varepsilon_x - \frac{k_x^2}{\omega\mu_0\mu}\right),$$

$$b_{42} = -b_{13}, \quad b_{43} = -i\left(\omega\mu_0\mu - \frac{k_x^2}{\omega\varepsilon_0\varepsilon_z}\right).$$

Coefficients (14) provide the basis for a study of the system of four first-order differential equations. These equations describe electromagnetic processes in inhomogeneous material media in which anisotropic properties are manifested in the process of propagation of electromagnetic harmonic time-dependent waves.

Matricant structure

The normalized solution of Eq. (13) is called matricant [23]. Maxwell's equations are then structured with application of the method of successive approximations and mathematical induction by comparing terms of the series for the direct and inverse matricants [23]:

$$T = E + \int_0^z B(z_1)dz_1 + \int_0^z \int_0^{z_1} B(z_1)B(z_2)dz_1 dz_2 + \dots,$$

$$T^{-1} = E - \int_0^z B dz_1 + \int_0^z \int_0^{z_1} B(z_2)B(z_1)dz_1 dz_2 - \dots$$

The structure of fundamental solutions of equations for electromagnetic wave propagation in cholesteric liquid crystals is defined by the structure of matrix coefficients (14). Comparing even and odd terms of these series, we finally obtain the matricant structures in the form

$$T = \begin{pmatrix} t_{11} & it_{12} & it_{13} & t_{14} \\ it_{21} & t_{22} & t_{23} & it_{24} \\ it_{31} & t_{32} & t_{33} & it_{34} \\ t_{41} & it_{42} & it_{43} & t_{44} \end{pmatrix},$$

$$T^{-1} = \begin{pmatrix} t_{22} & -it_{12} & it_{42} & -t_{32} \\ -it_{21} & t_{11} & -t_{41} & it_{31} \\ it_{42} & -t_{14} & t_{44} & -it_{34} \\ -t_{23} & -it_{13} & -it_{43} & t_{33} \end{pmatrix}.$$

The identity $TT^{-1} = T^{-1}T = E$ defines all invariant relations reflecting laws of conservation of electromagnetic waves during their propagation in inhomogeneous crystals:

$$t_{11}t_{22} + t_{12}t_{21} - t_{13}t_{24} - t_{14}t_{23} = 1,$$

$$it_{11}t_{42} - it_{12}t_{41} + it_{13}t_{44} - it_{14}t_{43} = 0,$$

$$t_{33}t_{44} + t_{34}t_{43} - t_{31}t_{42} - t_{32}t_{41} = 1,$$

$$-it_{41}t_{12} + it_{41}t_{11} - it_{43}t_{14} - it_{44}t_{13} = 0,$$

where t_{ij} are the elements of the direct matrix *T*.

Dispersion equations for an average medium

During propagation of long electromagnetic waves in cholesteric liquid crystals, $\lambda >> h$ ($h = 2\pi / q_0$), where λ is the wavelength and h is the spiral pitch or the inhomogeneity period of the cholesteric liquid crystal; then the matricant of Maxwell's equations can be written in the form [13]

$$T^{\pm} = \left[\frac{\hat{p} - \tilde{p}_2 E}{\tilde{p}_1 - \tilde{p}_2}\right]_{(2)} \left[E \cos \tilde{k} H \pm \frac{\langle B \rangle}{\tilde{k}} \sin \tilde{k} H\right] + \left[\frac{\hat{p} - \tilde{p}_1 E}{\tilde{p}_2 - \tilde{p}_1}\right]_{(2)} \left[E \cos \tilde{\chi} H \pm \frac{\langle B \rangle}{\tilde{\chi}} \sin \tilde{\chi} H\right]. \quad (15)$$

The matrix of the coefficients

$$\langle B \rangle = \frac{1}{h} \int_0^h B(Z) dZ \quad (16)$$

has been introduced into this relationship, where $B(Z)$ is given by Eq. (14), the matrix $\hat{p}$ in the second approximation is defined by the formula

$$\hat{p}_{(2)} = E + \frac{1}{2} \langle B \rangle^2 h^2, \quad (17)$$

and $\tilde{p}_{1,2}$ and $\tilde{k}, \tilde{\chi}$ are roots of the characteristic equation. This is defined by the condition

$$\det\left[\hat{p}_{(2)} - kE\right] = 0, \quad (18)$$

$$k_{1,2}^2 = \frac{1}{2}\left(b_{12}b_{21} - 2b_{13}b_{24} + b_{34}b_{43}\right) \pm \sqrt{\left(\frac{1}{2}b_{12}b_{21} - \frac{1}{2}b_{43}b_{34}\right)^2 - b_{12}b_{24}b_{13}b_{21} - b_{12}b_{24}^2 b_{43} - b_{13}^2 b_{21}b_{34} - b_{13}b_{34}b_{24}b_{43}}.$$

The matricant of Maxwell's equations describing the propagation of electromagnetic waves in the cholesteric crystal in the long-wavelength approximation ($\lambda >> 2\pi / q_0$) has the form

$$T^{\pm} = \left[\hat{\pi} + \frac{1}{2} E\right] \left[\cos \tilde{k} Z \pm \frac{\langle B \rangle}{\tilde{k}} \sin \tilde{k} Z\right] - \left[\hat{\pi} - \frac{1}{2} E\right] \left[\cos \tilde{\chi} Z \pm \frac{\langle B \rangle}{\tilde{\chi}} \sin \tilde{\chi} Z\right], \quad (19)$$

where the matrix $\hat{\pi}$ has the structure

$$\hat{\pi} = \begin{bmatrix} \pi_1 & 0 & 0 & \pi_{14} \\ 0 & \pi_1 & \pi_{23} & 0 \\ 0 & \pi_{14} & -\pi_1 & 0 \\ \pi_{23} & 0 & 0 & -\pi_1 \end{bmatrix},$$

$$\pi_{ij} = \left(\frac{p_{ij}}{2\Delta}\right).$$

Thus we have obtained the structure of the matricant of Maxwell's equations and the roots of dispersion equation (18).

Now we analyze the roots of the dispersion equation. Considering that the crystal parameters are periodic functions of the z coordinate ($\varphi(z) = \varphi(z + h)$ and $\psi(z) = \psi(z + h)$), we can determine the general type of electromagnetic wave dispersion from the structure of the matrix of fundamental solutions. Based on provisions obtained in [13], we can write the condition of derivation of dispersion equation (18)

$$\hat{p} = \frac{1}{2}\left[T + T^{-1}\right], \tag{20}$$

where

$$\hat{p} = \begin{pmatrix} p_1 & 0 & p_{13} & p_{14} \\ 0 & p_1 & p_{23} & p_{24} \\ p_{24} & -p_{14} & p_2 & 0 \\ -p_{23} & p_{13} & 0 & p_2 \end{pmatrix}.$$

The roots of characteristic equation (18) give the following formulas for the dispersion:

$$\cos k_1 z = \lambda_1,$$

$$\cos k_2 z = \lambda_2,$$

where λ_1 and λ_2 are roots of Eq. (18):

$$\begin{aligned} \lambda_{1,2} &= \frac{p_1 + p_2}{2} \pm \frac{1}{2}\sqrt{\left(p_1 + p_2\right)^2 - 4\left(p_{14}p_{23} - p_{13}p_{24}\right)}, \\ \lambda_{1,2} &= \frac{p_1 + p_2}{2} \pm \frac{1}{2}\sqrt{\left(p_1 - p_2\right)^2 - 4\left(p_{14}p_{23} - p_{13}p_{24}\right)}. \end{aligned} \tag{21}$$

Thus, dispersion equations (21) in the analytical form have been derived from the structure of the matricant of Maxwell's equations (6)–(9) and characteristic equation (18).

CONCLUSIONS

In this work we have studied the propagation of electromagnetic waves in liquid cholesteric crystals by the matricant method. Based on Maxwell's equations and material relationships, the system of the first order differential equations describing the propagation of electromagnetic harmonic waves in anisotropic liquid crystals was obtained. The matrix of the coefficients and the structure of the matricant were determined in the general case.

As a result of research taking into account periodic changes of the components of the dielectric permittivity tensor, the dispersion equations were obtained for electromagnetic waves propagating in cholesteric liquid crystals. The roots of these equations determine the propagation velocity and the coefficients of electromagnetic wave attenuation in cholesteric liquid crystals. The analytical matricant form was obtained in terms of the Chebyshev–Gegenbauer polynomials.

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