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Reflection of thermoelastic wave on the interface of isotropic half-space and tetragonal syngony anisotropic medium of classes 4, 4/*m* with thermomechanical effect

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The thermoelastic wave propagation in a tetragonal syngony anisotropic medium of classes 4, $4/m$ having heterogeneity along z axis has been investigated by employing matrizant method. This medium has an axis of second-order symmetry parallel to z axis. In the case of the fourth-order matrix coefficients, the problems of wave refraction and reflection on the interface of homogeneous anisotropic thermoelastic mediums are solved analytically.

Keywords: anisotropic medium, thermoelasticity, harmonic waves, dispersion

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1. Introduction

The theory of elasticity is concerned with the mutual interaction between the mechanical and thermal fields in elastic objects.[1,2] This theory has many important applications in various engineering disciplines, such as civil, mechanical, and nuclear engineering. It is based on the famous hypothesis of Fourier law of heat conduction, by which the temperature distribution is described through parabolic differential equations. According to the theory of elasticity, the thermal signal is propagated instantly throughout the object, which is physically impossible since a finite time is required for the signal propagation. In order to resolve this problem and account for the influence of thermal relaxation time, a modified version of Fourier law known as the generalized thermoelasticity has been proposed. In this modified theory, hyperbolic-type equations are used to predict the heat distribution, hence, the heat propagation in the condensed matter is considered as a wave rather than a diffusion process. In order to investigate the transmission phenomena of waves in anisotropic media with different mechanical and physical properties, matrizant method was employed.[3,4] However, the investigation of thermoelastic wave propagation is based on the simultaneous solution of motion equations and the exact solution of these motion equations can be obtained in integrable cases.[5–7]

The application of matrizant method to non-destructive testing and the wave propagation in thermoelastic media can be found in Ref. [8]. Meanwhile, the propagation of heat wave along an arbitrary axis in orthotropic thermoelastic plates have been investigated by normal modes expansion method.[9] The work in Ref. [9] was done by using the generalized theory of thermoelasticity with single thermal relaxation time. The generalized theory of thermoelasticity has also been used when free harmonic waves interact with media composed of different layers. In such a case, the investigation was made by a technique combining linear transformation and transfer matrix method. By using this technique, the solutions in the case with interaction between free harmonic waves and layered media can be obtained. The importance of these solutions is that they provide dispersion characteristics of multilayered media.

2. Basic equations and formulation

The study of the propagation of thermoelastic waves in anisotropic media is based on the simultaneous solution of equations of motion. The equations of thermal conductivity proposed by Fourier have the following form:[1]

$$\sigma_{ij,j} = \rho \ddot{U}_i, \tag{1}$$

$$\lambda_{ij}\frac{\partial \theta}{\partial x_j} = -q_i, \tag{2}$$

$$\frac{\partial q_i}{\partial x_i} = -\mathrm{i}\omega\beta_{ij}\varepsilon_{ij} - \mathrm{i}\omega\frac{c_\varepsilon}{T_0}\theta, \tag{3}$$

where σ_{ij} are the components of stress tensor, ρ is the density of medium, λ_{ij} are the components of the thermal conductivity tensor, q_i are the components of the heat flow vector, ω is the angular frequency, β_{ij} are the thermomechanical parameters of medium, ε_{ij} are the components of the tensor of small Cauchy deformation, c_ε is the heat capacity under constant deformation, and $\theta = T - T_0$ is the temperature augments compared with natural state temperature T_0. When deformation is small, $|\theta/T_0| \ll 1$.

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The relationship between stress and strain can be described by Duhamel–Neumann relation as

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - \beta_{ij}\theta. \tag{4}$$

Here, equations (1)–(4) show that the relationship between temperature and stress generated in mechanical process is a function of the thermal field and deformation whereas they are independent variables.

By using the method of separation of variables, equations (1)–(4) can be reduced to a system of ordinary differential equations (medium heterogeneity is assumed along the z axis, i.e., axis $z \parallel A_2$. where A_2 is the symmetry axis of second order) as

$$\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}x} = \boldsymbol{BW}, \tag{5}$$

where boundary condition column vector $\boldsymbol{W}$ is given as

$$\begin{aligned}\boldsymbol{W}(x,y,z,t) &= [u_z(z),\sigma_{zz},u_x(z),\sigma_{xz},u_y(z),\sigma_{yz},q_z,\theta]^t \\ &\quad \times \exp(\mathrm{i}\omega t - \mathrm{i}mx - \mathrm{i}ny),\end{aligned} \tag{6}$$

where $u_z(z)$, $u_x(z)$, and $u_y(z)$ represent the projections of displacement vector on the corresponding coordinates, i is an imaginary number, and $m = k_x$, $n = k_y$, showing x and y components of a wave vector $\boldsymbol{k}$, respectively, and the coefficient matrix is given as

$$\boldsymbol{B} = B\left[c_{ijkl}(z),\beta_{ij}(z),\theta,\omega,m,n,l\right]. \tag{7}$$

Here, the elements of coefficient matrix $\boldsymbol{B}$ in Eq. (7) contain the wave propagation information in the medium. In this paper, we analyzed the coefficient matrix $\boldsymbol{B}$ to determine the polarization of the waves and the relationship among them diverging with the influence of thermomechanical effect.

3. The problem of wave reflection

We consider the problem of thermoelastic wave reflection at the interface between isotropic and anisotropic half-space environment tetragonal syngony classes 4, $4/m$ with thermomechanical effect. Because of thermomechanical effects, bound thermoelastic waves propagate in the thermoelastic medium.

We suppose that the interface separating two media is $z = 0$ plane so that the axes of the Cartesian coordinate system coincide with the corresponding crystallographic axes as shown in Fig. 1. We also suppose that the thermal wave strikes at the interface of an isotropic medium, that is, the heat flux vector lies in the plane of incidence. The incidence plane is the plane that contains vectors perpendicular to the interface and heat flux vector.

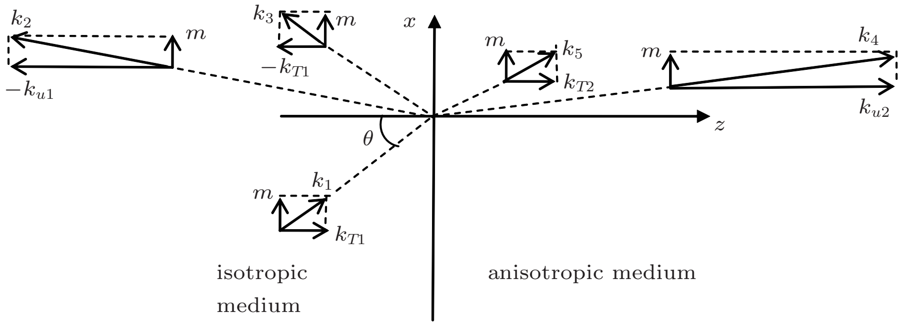

Fig. 1. Geometry of the problem. Here, m is the x component of a wave vector $\boldsymbol{k}$, while k_1, k_2, k_3, k_4, k_5 represent the projections of a wave vector on respective axes in isotropic and anisotropic media, k_{u1}, k_{u2}, k_{T1}, k_{T2} represent the projections of wave vector in isotropic and anisotropic media respectively. These projections depend on displacement vector $\boldsymbol{u}$ and heat flux vector $\boldsymbol{q}$.

In this case, the incident thermal wave in an anisotropic medium is related to the elastic longitudinal wave of z polarization, and equations (5) can be written as

$$\begin{cases}\dfrac{\mathrm{d}U_z}{\mathrm{d}Z} = \dfrac{1}{c_{33}}\sigma_{zz} + \dfrac{2\beta_{13}+\beta_{33}}{c_{33}}\theta, \\ \dfrac{\mathrm{d}\sigma_{zz}}{\mathrm{d}Z} = -\rho\omega^2 U_z, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}z} = -\dfrac{1}{\lambda_{33}}q_z, \\ \dfrac{\mathrm{d}q_z}{\mathrm{d}Z} = -\mathrm{i}\omega\dfrac{2\beta_{13}+\beta_{33}}{c_{33}}\sigma_{zz} - \mathrm{i}\omega\left(\dfrac{c_\varepsilon}{0} + \dfrac{\beta_{33}^2}{c_{11}}\right)\theta_z,\end{cases} \tag{8}$$

which can also be written in a matrix form as

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}z} = \boldsymbol{B}_2\boldsymbol{w}, \tag{9}$$

where $\boldsymbol{w} = (u_y,\sigma_{yz},\theta,q_z)^t$ and

$$\boldsymbol{B}_2 = \begin{pmatrix} 0 & b_{12} & b_{17} & 0 \\ b_{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{78} \\ 0 & -\mathrm{i}\omega b_{17} & b_{87} & 0 \end{pmatrix} \tag{10}$$

is the coefficient matrix of the second medium. The components of $\boldsymbol{B}_2$ have the following form:

$$b_{12}=\frac{1}{c_{33}},\quad b_{17}=\frac{(2\beta_{13}+\beta_{33})}{c_{33}},$$
$$b_{21}=-\omega^2\rho,\quad b_{87}=-\mathrm{i}\omega\left(\frac{\beta_{33}^2}{c_{11}}+\frac{c_\varepsilon}{T_0}\right),$$
$$b_{78}=-\frac{1}{\lambda_{33}}.$$

The matrizant of the second medium (direct wave) can be written as

$$T_2^+(0)=\frac{1}{2}(E+\mathrm{i}\alpha R_2),\tag{11}$$

with

$$\alpha=\frac{1}{k_{T2}k_{u2}(k_{T2}+k_{u2})}.\tag{12}$$

Here, k_{u2} and k_{T2} represent z components of the wave vector in the second medium. For the matrix coefficients $\boldsymbol{B}_2$ given in Eq. (10), we write

$$k_{u2}=\sqrt{\frac{1}{2}\left(-b_{12}b_{21}-b_{78}b_{87}-(b_{12}b_{21}-b_{78}b_{87})\sqrt{1-\frac{4\mathrm{i}\omega b_{17}^2b_{21}b_{78}}{(b_{12}b_{21}-b_{78}b_{87})^2}}\right)},\tag{13}$$

$$k_{T2}=\sqrt{\frac{1}{2}\left(-b_{12}b_{21}-b_{78}b_{87}+(b_{12}b_{21}-b_{78}b_{87})\sqrt{1-\frac{4\mathrm{i}\omega b_{17}^2b_{21}b_{78}}{(b_{12}b_{21}-b_{78}b_{87})^2}}\right)}.\tag{14}$$

Whereas we define

$$a=-b_{12}b_{21}-b_{78}b_{87},\tag{15}$$
$$\Delta=(b_{12}b_{21}-b_{78}b_{87})\sqrt{1-\frac{4\mathrm{i}\omega b_{17}^2b_{21}b_{78}}{(b_{12}b_{21}-b_{78}b_{87})^2}},\tag{16}$$

and introduce a matrix $\boldsymbol{R}_2$ for the matrix coefficients $\boldsymbol{B}_2$ given in Eq. (10) and as a result, we obtain

$$\boldsymbol{R}_2=\begin{pmatrix}0 & r_{12} & r_{13} & 0\\ r_{21} & 0 & 0 & r_{24}\\ -\mathrm{i}\omega r_{24} & 0 & 0 & r_{34}\\ 0 & -\mathrm{i}\omega r_{13} & r_{43} & 0\end{pmatrix},\tag{17}$$

with

$$r_{12}=-b_{78}(\mathrm{i}\omega b_{17}^2+b_{12}b_{87})-b_{12}\sqrt{b_{21}b_{78}(\mathrm{i}\omega b_{17}^2+b_{12}b_{87})},$$
$$r_{13}=b_{17}\sqrt{b_{21}b_{78}(\mathrm{i}\omega b_{17}^2+b_{12}b_{87})},$$
$$r_{21}=-b_{21}b_{78}b_{87}+b_{21}\sqrt{d_{21}b_{78}(\mathrm{i}\omega b_{17}^2+b_{12}b_{87})},$$
$$r_{24}=b_{17}b_{21}b_{78},$$
$$r_{34}=-b_{12}b_{21}b_{78}+b_{78}\sqrt{b_{21}b_{78}(\mathrm{i}\omega b_{17}^2+b_{12}b_{87})},$$
$$r_{43}=-b_{21}(\mathrm{i}\omega b_{17}^2+b_{12}b_{87})+b_{87}\sqrt{b_{21}b_{78}(\mathrm{i}\omega b_{17}^2+b_{12}b_{87})}.$$

In an isotropic medium, the incident thermal wave is not related to the elastic properties of medium, so the coefficient matrix of the first medium $\boldsymbol{B}_1$ takes the form

$$\boldsymbol{B}_1=\begin{pmatrix}0 & b_{12} & 0 & 0\\ b_{21} & 0 & 0 & 0\\ 0 & 0 & 0 & b_{78}\\ 0 & 0 & b_{87} & 0\end{pmatrix},\tag{18}$$

with

$$b_{12}=\frac{2}{c_{11}-c_{12}},\quad b_{21}=-\rho_1\omega^2+\frac{m^2(c_{11}-c_{12})}{2},$$
$$b_{78}=-\frac{1}{\lambda_{11}},\quad b_{87}=-\frac{\mathrm{i}\omega c_\varepsilon}{T_0}.$$

As is seen from matrix (18), the coefficient matrix $\boldsymbol{B}_1$ can be divided into two matrices of second order, so matrizant of the first medium, i.e., the isotropic medium, can be written as

$$\boldsymbol{T}^{\pm}{}_{\text{ycp}}=\frac{1}{2}\left(\boldsymbol{E}\mp\frac{\langle\boldsymbol{B}\rangle}{\mathrm{i}k}\right)\mathrm{e}^{\mp\mathrm{i}kz}.\tag{19}$$

As a result, we obtain

$$\boldsymbol{T}_1^{\pm}=\begin{pmatrix}1 & \pm\frac{\mathrm{i}b_{12}}{k_{u1}} & 0 & 0\\ \pm\frac{\mathrm{i}b_{21}}{k_{u1}} & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\frac{\mathrm{e}^{\mp\mathrm{i}k_{u1}z}}{2}+\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & \pm\frac{\mathrm{i}b_{78}}{k_{T1}}\\ 0 & 0 & \pm\frac{\mathrm{i}b_{87}}{k_{T1}} & 1\end{pmatrix}\frac{\mathrm{e}^{\mp\mathrm{i}k_{T1}z}}{2},\tag{20}$$

with

$$k_{u1}=\sqrt{-b_{12}b_{21}},\quad k_{T1}=\sqrt{-b_{78}b_{87}},\tag{21}$$

representing the z-component wave vectors of the first medium.

Substituting the matrizant of the second medium (Eq. (11)), Eq. (17), and the matrizant of the first medium (Eq. (20)) into a matrix $\boldsymbol{G}$, we obtain

$$\boldsymbol{G}=\begin{pmatrix}g_{11} & 0 & 0 & g_{14}\\ 0 & g_{22} & g_{23} & 0\\ 0 & g_{32} & g_{33} & 0\\ g_{41} & 0 & 0 & g_{44}\end{pmatrix},\tag{22}$$

with

$$g_{11}=-1+\frac{2b_{21}(b_{78}+k_{T1}r_{34}\alpha)}{\Delta_1},\quad g_{14}=-\frac{2b_{78}k_{u1}r_{24}\alpha}{\Delta_1},$$
$$g_{22}=-1+\frac{2b_{12}(b_{87}+k_{T1}r_{43}\alpha)}{\Delta_2},$$
$$g_{23}=-\frac{2b_{87}k_{u1}r_{13}\alpha}{\Delta_2},\quad g_{32}=\frac{2\mathrm{i}\omega b_{12}k_{T1}r_{13}\alpha}{\Delta_2},$$
$$g_{33}=-1+\frac{2b_{87}(b_{12}+k_{u1}r_{12}\alpha)}{\Delta_2},$$
$$g_{41}=\frac{2\mathrm{i}\omega b_{21}k_{T1}r_{24}\alpha}{\Delta_1},\quad g_{44}=-1+\frac{2b_{78}(b_{21}+k_{u1}r_{21}\alpha)}{\Delta_1},$$
$$\Delta_1=b_{21}(b_{78}+k_{T1}r_{34}\alpha)+k_{u1}\alpha(b_{78}r_{21}+k_{T1}\alpha(\mathrm{i}\omega r_{24}^2+r_{21}r_{34})),$$
$$\Delta_2=b_{12}(b_{87}+k_{T1}r_{43}\alpha)+k_{u1}\alpha(b_{43}r_{12}+k_{T1}\alpha(\mathrm{i}\omega r_{13}^2+r_{12}r_{43})).$$

The amplitude of incident wave vector can be written as

$$\boldsymbol{w}_0=(0,0,\theta_0,q_0)^t, \tag{23}$$

which relates the amplitudes of the temperature increment θ_0 and heat flux q_0 of the incident thermal wave field and

$$\theta_0=\frac{\mathrm{i}b_{78}}{k_{T1}}q_0\quad\text{or}\quad\theta_0=\frac{k_{T1}}{\mathrm{i}b_{87}}q_0. \tag{24}$$

According to Eqs. (22) and (23), the amplitudes of reflected and refracted wave vectors can be written as

$$\begin{cases}u_{\mathrm{r}}=g_{14}q_0,\\ \sigma_{\mathrm{r}}=g_{23}\theta_0,\\ \theta_{\mathrm{r}}=g_{33}\theta_0,\\ q_{\mathrm{r}}=g_{44}q_0,\end{cases} \tag{25}$$

$$\begin{cases}u_{\mathrm{t}}=g_{14}q_0,\\ \sigma_{\mathrm{t}}=g_{23}\theta_0,\\ \theta_{\mathrm{t}}=(1+g_{33})\theta_0,\\ q_{\mathrm{t}}=(1+g_{44})q_0.\end{cases} \tag{26}$$

It can be seen from expressions (25) and (26) that due to the incident thermal wave, $u_{\mathrm{r}}=u_{\mathrm{t}}$ and $\sigma_{\mathrm{r}}=\sigma_{\mathrm{t}}$.

Similarly, according to Eqs. (19) and (20), the amplitudes of displacement and stress of reflected waves are given as

$$\boldsymbol{w}_{\mathrm{refl}}(0)=T_1^-(0)\boldsymbol{w}_{\mathrm{r}}=\boldsymbol{w}_{\mathrm{r}}, \tag{27}$$
$$\boldsymbol{w}_{\mathrm{refr}}(0)=T_2^+(0)\boldsymbol{w}_{\mathrm{t}}=\boldsymbol{w}_{\mathrm{t}}, \tag{28}$$

and the amplitudes of temperature and heat flux of reflected and refracted waves are given as

$$\sigma_{\mathrm{r}}=-\frac{k_{u1}}{\mathrm{i}b_{12}}u_{\mathrm{r}}\quad\text{or}\quad\sigma_{\mathrm{r}}=-\frac{\mathrm{i}b_{21}}{k_{u1}}u_{\mathrm{r}}, \tag{29}$$

$$\theta_{\mathrm{r}}=-\frac{\mathrm{i}b_{78}}{k_{T1}}q_{\mathrm{r}}\quad\text{or}\quad q_{\mathrm{r}}=-\frac{k_{T1}}{\mathrm{i}b_{87}}q_{\mathrm{r}}, \tag{30}$$

$$\begin{cases}u_{\mathrm{t}}=\mathrm{i}\alpha(r_{12}\sigma_{\mathrm{t}}+r_{13}\theta_{\mathrm{t}}),\\ \sigma_{\mathrm{t}}=\mathrm{i}\alpha(r_{21}u_{\mathrm{t}}+r_{24}q_{\mathrm{t}}),\\ \theta_{\mathrm{t}}=\mathrm{i}\alpha(-\mathrm{i}\omega r_{24}u_{\mathrm{t}}+r_{34}q_{\mathrm{t}}),\\ q_{\mathrm{t}}=\mathrm{i}\alpha(-\mathrm{i}\omega r_{13}\sigma_{\mathrm{t}}+r_{43}\theta_{\mathrm{t}}).\end{cases} \tag{31}$$

According to the matrizant of the first medium $\boldsymbol{B}_1$ (Eq. (19)) and matrix (20), for the incident wave $\boldsymbol{w}_{\mathrm{nao}}=T_1^+\boldsymbol{w}_0$ and $\boldsymbol{w}_{\mathrm{omp}}=T_1^-\boldsymbol{w}_{\mathrm{r}}$ for the reflected wave, the incident heat wave, reflected elastic, and thermal waves can be written in explicit forms as

$$\begin{cases}\theta_z^{\mathrm{nao}}=\theta_0\,\mathrm{e}^{-\mathrm{i}k_{T1}z},\\ q_z^{\mathrm{nao}}=q_0\,\mathrm{e}^{-\mathrm{i}k_{T1}z},\end{cases} \tag{32}$$

$$\begin{cases}u_z^{\mathrm{omp}}=g_{14}q_0\,\mathrm{e}^{\mathrm{i}k_{u1}z},\\ \sigma_{zz}^{\mathrm{omp}}=g_{23}\theta_0\,\mathrm{e}^{\mathrm{i}k_{u1}z},\\ \theta_z^{\mathrm{omp}}=g_{33}\theta_0\,\mathrm{e}^{\mathrm{i}k_{T1}z},\\ q_z^{\mathrm{omp}}=g_{44}q_0\,\mathrm{e}^{\mathrm{i}k_{T1}z}.\end{cases} \tag{33}$$

According to Eqs. (8), (35), and (33), we can calculate the energy fluxes of the reflected elastic and thermal waves. Finally, the flow of heat energy is given by

$$q_{\mathrm{T}}=\theta\boldsymbol{q}, \tag{34}$$

while the flow of elastic energy is given by

$$P_j=-\sigma_{ij}\frac{\partial u_i}{\partial t}. \tag{35}$$

4. Conclusion

In this paper, based on the matrizant method,[4] the propagation of thermoelastic waves in an anisotropic medium tetragonal syngony of classes 4, $4/m$, in the case of inhomogeneity along z axis has been studied. We have analytically solved the problem of reflection and refraction at the boundary of the homogeneous anisotropic thermoelastic media, in the case of coefficient matrices having order 4. It is expected that our results will be useful for future experiments in various branches of civil and mechanical engineering.

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