ISSN 2518-1483 (Online),
ISSN 2224-5227 (Print)

2023 • 1

ҚАЗАҚСТАН РЕСПУБЛИКАСЫ
ҰЛТТЫҚ ҒЫЛЫМ АКАДЕМИЯСЫНЫҢ

БАЯНДАМАЛАРЫ

ДОКЛАДЫ

НАЦИОНАЛЬНОЙ АКАДЕМИИ НАУК
РЕСПУБЛИКИ КАЗАХСТАН

REPORTS

OF THE NATIONAL ACADEMY OF SCIENCES
OF THE REPUBLIC OF KAZAKHSTAN

PUBLISHED SINCE JANUARY 1944

ALMATY, NAS RK

REPORTS OF THE NATIONAL ACADEMY OF SCIENCES OF THE REPUBLIC OF KAZAKHSTAN
ISSN 2224-5227
Volume 345, Number 1 (2023), 108–124
https://doi.org/10.32014/2023.2518-1483.188

UDC 336.77, 338.43(477)

© E.B. Arinov[1], L.R. Kundakova[1*], N.A. Ispulov[2], A.K. Seitkhanova[3], A.Zh. Zhumabekov[2], 2023

[1]O.A. Baikonyrov Zhezkazgan University, Kazakhstan;
[2]Toraighyrov University, Pavlodar, Kazakhstan;
[3]Pavlodar Pedagogical University, Pavlodar, Kazakhstan.
E-mail: nurlybek_79@mail.ru

THE SOLUTION OF DIFFERENTIAL EQUATIONS FOR ELASTIC DISTURBANCES IN THE CYLINDRICAL COORDINATE SYSTEM WITH REGARD TO THE INERTIAL COMPONENTS

Arinov E.B. — Doctor of Physical-mathematical Science. Professor. O.A. Baikonyrov Zhezkazgan University. Kazakhstan
E-mail: arinov91@mail.ru;
Kundakova L.R. — Candidate of Science (Economics), Associate Professor. Department of Economics and Law. O.A. Baikonyrov Zhezkazgan University, Kazakhstan
E-mail: kundakova@mail.ru;
Ispulov N.A. — Candidate of Physical-mathematical Science, Associate Professor. Faculty of Computer Science. Toraighyrov University, Pavlodar, Kazakhstan
E-mail: nurlybek_79@mail.ru. ORCID: 0000-0003-4703-1413;
Seitkhanova A.K. — PhD. Associate Professor, Higher School of Natural Sciences. Pavlodar, Kazakhstan
E-mail: ainur1179@mail.ru. ORCID: 0000-0002-6667-4548;
Zhumabekov A.Zh. — PhD. Associate Professor, Faculty of Computer Science. Toraighyrov University, Pavlodar, Kazakhstan
E-mail: almar89-89@mail.ru. ORCID: 0000-0003-2360-3747.

Abstract. In this review article, an internal boundary value problem is solved for perturbations in a cylindrical coordinate system considering the inertial terms. The equation in displacements presented in the Tedone form are solved for the case of compressible material of cylindrical bodies. Solutions of homogeneous differential equations are obtained in Bessel functions with preliminary separation of variables of inhomogeneous equations, which are found by the method of undetermined coefficients. Presented in the form of Tedone, the equation in displacements is solved under the conditions of compressibility of cylindrical material bodies. In this case, the volume expansion obeys the types of the Helmholtz equation. Solutions of homogeneous differential equations in Bessel functions with preliminary separation of variables of inhomogeneous equations are constructed in the work.

These homogeneous differential equations are found by the method of indefinite coefficients. Also in this review article, the internal boundary value problem of perturbation is solved. The displacement equation, expressed in the Tedone form, is solved for the compressed material of some cylindrical bodies. The solution of homogeneous differential equations in Bessel functions with preliminary separation of variables is obtained. It should be noted that the solution of inhomogeneous differential equations is obtained by the method of indefinite coefficients.

Key words: cylindrical coordinate system, differential equations, solutions of heterogeneous equations, Tedone forms, Bessel functions

© Е.Б. Аринов[1], Л.Р. Кундакова[1*], Н.А. Испулов[2], А.К. Сейтханова[3], А.Ж. Жумабеков[2], 2023

[1]О.А. Байконыров атындағы Жезказган Университеті, Казахстан;

[2]Торайғыров Университеті, Павлодар, Казахстан;

[3]Павлодар Педагогикалық Университеті, Павлодар, Казахстан.

E-mail: nurlybek_79@mail.ru

ЦИЛИНДРЛІК КООРДИНАТАЛАР ЖҮЙЕСІНДЕ ИНЕРЦИЯЛЫҚ ҚОСЫЛҒЫШТАРДЫ ЕСКЕРЕ ОТЫРЫП, СЕРПІМДІ АУЫТҚУЛАР ҮШІН ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕРДІ ШЕШУ

Аринов Е.Б. — физика-математика ғылымдарының докторы, профессор. О.А. Байқоңыров атындағы Жезқазған университеті, Қазақстан

E-mail: arinov91@mail.ru;

Құндақова Л.Р. — ғылым кандидаты (экономика), экономика және құқық кафедрасының доценті, О.А. Байқоңыров атындағы Жезқазған университеті, Қазақстан

E-mail: kundakova@mail.ru;

Испулов Н.А. — физика-математика ғылымдарының кандидаты, доцент. Торайғыров университеті, компьютерлік ғылымдар факультеті, Павлодар, Қазақстан

E-mail: nurlybek_79@mail.ru. ORCID: 0000-0003-4703-1413;

Сейтханова А.Қ. — PhD докторы. Жаратылыстану ғылымдары жоғары мектебінің доценті. Павлодар, Қазақстан

E-mail: ainur1179@mail.ru. ORCID: 0000-0002-6667-4548;

Жұмабеков А.Ж. — PhD докторы, доцент, компьютерлік ғылымдар факультеті. Торайғыров университеті, Павлодар, Қазақстан

E-mail: almar89-89@mail.ru. ORCID: 0000-0003-2360-3747.

Аннотация. Бұл шолу мақаласында инерциялық мүшелерді ескере отырып, цилиндрлік координаттар жүйесіндегі қозулар үшін ішкі шекаралықесеп шешіледі. Цилиндрлік денелердің сығылатын материалы үшін Тедоне түріндегі қозғалыстардың теңдеулері шешіледі. Анықталмаған коэффициенттер әдісі арқылыанықталатын біртекті емес теңдеулердің айнымалыларын алдын-ала ажыратумен Бессель функцияларындағы біртекті дифференциалдық теңдеулердің шешімдері алынады. Тедоне түріндегі қозғалыстардың теңдеуі цилиндрлік материалдық денелердің сығылу жағдайында шешіледі. Бұл жағдайда көлемдік кеңею Гельмгольц теңдеулеріне бағынады. Жұмыста біртекті емес теңдеулердің айнымалылары алдын-ала ажыратылатын Бессель функцияларындағы біртекті дифференциалдық теңдеулердің шешімдері құрылады. Бұл біртекті дифференциалдық теңдеулер анықталмаған коэффициенттер әдісімен табылады. Сондай-ақ, осы шолу мақаласында қозудың ішкі шекаралық есебі шешіледі. Тедоне түріндегі қозғалыстың теңдеуі кейбір цилиндрлік денелердің сығылған материалы үшін шешіледі. Бессель функцияларындағы біртекті дифференциалдық теңдеулердің шешімі айнымалыларын алдын-ала ажыратумен алынды. Біртекті емес дифференциалдық теңдеулерді шешу анықталмаған коэффициенттер әдісімен алынатындығын айта кету керек.

Түйін сөздер: цилиндрлік координаталар жүйесі, дифференциалдық теңдеулер, біртекті емес теңдеулердің шешімдері, Тедоне түрі, Бессель функциясы

© Е.Б. Аринов[1], Л.Р. Кундакова[1*], Н.А. Испулов[2], А.К. Сейтханова[3], А.Ж. Жумабеков[2], 2023

[1]Жезказганский университет имени О.А. Байконырова, Казахстан;
[2]Торайгыров Университет, Павлодар, Казахстан;
[3]Павлодарский педагогический университет, Павлодар, Казахстан.
E-mail:nurlybek_79@mail.ru

РЕШЕНИЕ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ДЛЯ УПРУГИХ ВОЗМУЩЕНИЙ В ЦИЛИНДРИЧЕСКОЙ СИСТЕМЕ КООРДИНАТ С УЧЕТОМ ИНЕРЦИАЛЬНЫХ СОСТАВЛЯЮЩИХ

Аринов Е.Б. — доктор физико-математических наук, профессор. Жезказганский университет имени О.А. Байконурова, Казахстан
E-mail: arinov91@mail.ru;
Кундакова Л.Р. — кандидат экономических наук, доцент кафедры экономики и права. Жезказганский университет имени О.А. Байконурова, Казахстан
E-mail: kundakova@mail.ru;
Испулов Н.А. — кандидат физико-математических наук, доцент, факультет компьютерных наук Торайгыров Университета, г. Павлодар, Казахстан
E-mail: nurlybek_79@mail.ru. ORCID: 0000-0003-4703-1413;
Сейтханова А.К. — PhD, доцент Высшей школы естественных наук, г. Павлодар, Казахстан
E-mail: ainur1179@mail.ru. ORCID: 0000-0002-6667-4548;
Жумабеков А.Ж. — PhD, ассоц. профессор факультета компьютерных наук Торайгыров Университета. г. Павлодар, Казахстан
E-mail: almar89-89@mail.ru. ORCID: 0000-0003-2360-3747.

Аннотация. В данной обзорной статье решается внутренняя краевая задача для возмущений в цилиндрической системе координат с учетом инерционных членов. Уравнения в перемещениях, представленные в форме Тедоне, решаются для случая сжимаемого материала цилиндрических тел.

Получены решения однородных дифференциальных уравнений в функциях Бесселя с предварительным разделением переменных неоднородных уравнений, которые находятся методом неопределенных коэффициентов. Представленное в виде Тедоне уравнение в перемещениях решается в условиях сжимаемости цилиндрических материальных тел. В этом случае объемное расширение подчиняется уравнениям Гельмгольца. В работе строятся решения однородных дифференциальных уравнений в функциях Бесселя с предварительным разделением переменных неоднородных уравнений. Эти однородные дифференциальные уравнения находятся методом неопределенных коэффициентов. Также в данной обзорной статье решается внутренняя краевая задача возмущения. Уравнение перемещений, выраженное в форме Тедоне, решается для сжатого материала некоторых цилиндрических тел. Получено решение однородных дифференциальных уравнений в функциях Бесселя с предварительным разделением переменных. Следует отметить, что решение неоднородных дифференциальных уравнений получается методом неопределенных коэффициентов.

Ключевые слова: цилиндрическая система координат, дифференциальные уравнения, решения неоднородных уравнений, форма Тедоне, функция Бесселя

Introduction

Explosions or earthquakes are impulsive external influences. If the duration of the action of an external force of the pulse type is small compared to the period of its own oscillations, then after the expiration of the oscillation force of the system that has left the equilibrium position with an initial velocity equal to the magnitude of the external force pulse.

It is assumed that the impulse effect on the system occurs in some sufficiently large neighborhood of the location of the mine in order not to produce a direct destructive effect on the environment that includes this mine. As it is known /1/, as waves move away from the initial disturbance area, their amplitude, if the effects of dispersion are neglected, changes inversely proportional to the square root of the distance (surface waves) or inversely proportional to the distance (volume waves) from the source of the initial disturbance (pulse). It seems possible to apply the linear theory of elastic dynamic oscillations for the formulated problem, as is customary in theoretical seismology.

The relevance of studying the laws of propagation of electromagnetic and elastic (mechanical) waves is associated with the presence of mutual transformation. Wave processes in coupled fields reflect the mutual influence of elastic, electromagnetic and thermal fields. In (Loran et al., 2020), the process of scattering of electromagnetic plane waves by an ideal half-plane of an electric conductor in an anisotropic medium is studied. The method used to solve the problem is the transition boundary method.

Research in the field of elastic waves is considered in the works (Silva et al., 2021) as one of the most important conditions for solving boundary value problems

(Li et al., 2021). Various methods of solutions and wave propagation are presented, for example, symmetric and antisymmetric waves under two thermal conditions Nowacki, wave propagation in three-dimensional space Sharma, wave propagation in elastic media by the matricant method and et.al. (Dossumbekov et al., 2021: 8; Kurmanov et al., 2020: 085505; Ispulov et al., 2017: 5236898; Mussa et al., 2022: 12; Nowacki et al., 1986; Sharma et al., 2004: 15; Ispulov et al., 2022: 4; Ispulov et al., 2022: 11; Tleukenov, 2004; Tleukenov, 2019).

The boundary problems of disturbances in the cylindrical coordinate system with regard to inertial components were solved. This information is necessary to study the free elastic vibrations of the system. The equation in movements, presented in the form of the Tedone, are decided for the case of the compressible material of cylindrical bodies. The volume expansion is subject to the Helmholtz equation. Solutions of homogeneous differential equations are obtained in non-cells functions with preliminary division of variables. Solutions of heterogeneous differential equations are found by the method of undefined coefficients. The boundary conditions of free oscillations at the border for internal and external boundary problems are written down.

Research Material and methods

A type of problem in the theory of elasticity associated with the propagation of oscillations or stationary states of vibration in an elastic medium. In the simplest case, but also the most important in practical applications — the linear theory of homogeneous isotropic elastic bodies — such problems can be reduced to finding a solution to the Lamé equation. These velocities are the displacement velocities of two types of deformations in a linearly elastic isotropic body. It can also be shown that, under certain conditions, surface waves can propagate along interfaces and that they have characteristic propagation velocities (Rayleigh waves on a free surface, Stoneley waves on an elastic medium boundary).

The cases of the appearance of discontinuities in the first derivatives of the displacement with respect to the characteristics (strong discontinuity) are also investigated. If the characteristic jumps affect only the component of the gradient normal vector, and the tangential components of this vector and the displacements themselves remain continuous, then the discontinuity is called a constant force discontinuity. In this case, the conditions of kinematic and dynamic compatibility are satisfied on the characteristic surface, which play an important role in solving dynamic problems by the method of characteristics.

The action of dynamic deformations of an elastic body becomes more complicated if this body has a finite boundary. Each point of such a boundary, in contact with any one of the perturbations propagating from fronts that are complex in themselves, causes at least two new types of deformation.

The main ones are the following types of boundary value problems: the first shows displacements; secondly, voltages are shown; the third one shows linear combinations of displacements and stresses; fourthly, the normal component of the shear and the tangential components of the tension are shown; fifth, the tangential components of the displacement and the normal component of the stress are shown; Sixth, the displacements are shown in one part of S, and the tensions are shown in the appendix.

In contrast to the Cauchy problem, which is completely solvable in the general case, solutions to mixed problems are obtained only in special cases. The most important of them are: closed solutions of the first and second main mixed problems for a half-plane or half-space, obtained by the method of complex waves and a generalization of the method of characteristics; solutions of the wave equation for the sphere obtained by the method of functionally invariant integrals; solution of some problems of the theory of elasticity by generalizing this method; and solving a number of diffraction problems. As a rule, decisions cannot be made in private; however, very general results can be obtained using the methods of potential theory and the theory of singular integral equations.

The methods of geometric optics are also used in the case of surface waves. The boundary condition of zero surface tension can be satisfied by the superposition of longitudinal and transverse waves with complex eikonals. Such constructions give rise to a wide class of surface waves, a striking example of which are Rayleigh waves.

The geometric-optical theory can also be developed for other types of surface waves: for waves similar to Love waves and for the so-called surface trapped waves. An analogue of the considered Love waves are stationary high-frequency waves, the phase velocity of which is close to the velocity of transverse waves, and the direction of the displacement vector, according to the first approximation, is the frequency normal to the surface and the direction of wave propagation. Waves captured by the surface also have a surface velocity close to that of transverse waves, but their polarization is different — the displacement vector lies in the plane formed by the normal to the surface and the direction of wave propagation.

The equation of dynamic oscillations of the elastic body of the vector form has

$$\frac{1}{1-2\nu}\, grad\, div\, \bar{u}_* + \nabla^2 \bar{u}_* = \frac{\rho}{\mu}\ddot{\bar{u}}_* \qquad (1)$$

Where the Laplace operator in the cylindrical coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \qquad (2)$$

$\bar{u}_*$ is the displacement vector, μ is the shear module, v is Poisson ratio, p – Density of area`s material; differentiation twice in time are marked by two points above the vector, $\bar{u}_* t$; r; φ; z – cylindrical coordinates, and r – radius, φ – azimuth angle, z - coordinate in the direction of the axis of the solid cylindrical body. The vector of displacements is represented as:

$$\bar{u}_*(t,\varphi,z) = \exp(ift)\bar{u}_{**}(r,\varphi,z), \qquad (3)$$

where i – imaginary unit, f – complex frequency.

With regard to (3) equation (1) acquires the following form:

$$\frac{1}{1-2\nu}\, grad\, div\, \bar{u}_{**} + \nabla^2 \bar{u}_{**} = \frac{\rho}{\mu} f^2 \bar{u}_{**}. \qquad (4)$$

Let the volumetric extension obey the Helmholtz equation. (Lurie, 2005) to the next of the equation (4):

$$\nabla^2 \Theta_{**} + \frac{\rho}{2\mu}\frac{1-2\nu}{1-\nu} f^2 \Theta_{**} = 0, \qquad (5)$$

where

$$\Theta_{**} = \frac{\partial u_{**}}{\partial r} + \frac{u_{**}}{r} + \frac{\partial v_{**}}{r\partial\varphi} + \frac{\partial w_{**}}{\partial z}, \qquad (6)$$

u_{**}, v_{**}, w_{**} – are the components of the vector $\bar{u}_{**}$.

From equation (1) follows the equation in the movements in the form of Thetone:

$$\nabla^2\left[\bar{u}_{**} + \frac{1}{2(1-2\nu)} \bar{r} f^2 \Theta_{**}\right] + \frac{\rho}{\mu}\left[f^2 \bar{u}_{**} + \frac{1}{4(1-\nu)} \bar{r} f^2 \Theta_{**}\right] = 0, \qquad (7)$$

where $\bar{r}$ – is the radius-vector of the arbitrary point of the body

$$\bar{r} = r\bar{e}_r + z\bar{k}, \qquad (8)$$

$\bar{e}_r$, are basic vectors.

Let`s rewrite the equation (7) in the form of:

$$\left(\nabla^2 + \frac{\rho}{\mu} f^2\right)\bar{u}_{**} = -\frac{\rho}{\mu}\frac{1}{4(1-\nu)} \bar{r} f^2 \Theta_{**} - \frac{1}{2(1-2\nu)} \nabla^2 (\bar{r} f^2 \Theta_{**}). \qquad (9)$$

Producing the specified differential operations in the equation (9) and recording it in the coordinate form, we get the following system of differential equations in the private productively unknown components, displacements:

$$\left(\nabla^2 + \frac{1}{r^2} + \frac{\rho}{\mu} f^2\right) u_{**} - \frac{2}{r^2}\frac{\partial v_{**}}{\partial\varphi} = -\frac{\rho}{\mu}\frac{1}{4(1-\nu)} r f^2 \Theta_{**} - \frac{1}{2(1-2\nu)}\left(\nabla^2 + \frac{2}{r}\frac{\partial}{\partial r}\right)\Theta_{**};$$

$$\left(\nabla^2 + \frac{\rho}{\mu} f^2 - \frac{1}{r^2}\right) v_{**} + \frac{2}{r^2}\frac{\partial u_{**}}{\partial\varphi} = -\frac{1}{1-2\nu}\frac{1}{r}\frac{\partial\Theta_{**}}{\partial\varphi};$$

$$\left(\nabla^2 + \frac{\rho}{\mu} f^2\right) w_{**} = -\frac{\rho}{\mu}\frac{z f^2 \Theta_{**}}{4(1-\nu)} - \frac{1}{2(1-2\nu)}\left(\nabla^2 + \frac{2}{z}\frac{\partial}{\partial z}\right)\Theta_{**}; \qquad (10)$$

The right parts of the equations (10) are the known functions found in this paragraph from the general Helmoltz equation (5).

Considering (5), representing displacement, and volumetric expansion in the form of:

$$u_{**}(r,\varphi,z) = u(r)\sin\lambda z\cos n\varphi, \quad v_{**}(r,\varphi,z) = v(r)\sin\lambda z\cos n\varphi, \tag{11}$$

$$w_{**}(r,\varphi,z) = w(r)\cos\lambda z\cos n\varphi, \quad \Theta_{**}(r,\varphi,z) = \Theta(r)\sin\lambda z\cos n\varphi$$

where $n = 0,1,2,\ldots$ (12)

From (10) we get:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\lambda^2-\frac{n^2+1}{r^2}+\frac{\rho}{\mu}f^2\right)u(r)-\frac{2n}{r^2}v(r)=-\frac{1}{1-2v}\frac{d\Theta(r)}{dr},$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\lambda^2-\frac{n^2+1}{r^2}+\frac{\rho}{\mu}f^2\right)v(r)-\frac{2n}{r^2}u(r)=\frac{n}{1-2v}\frac{1}{r}\Theta(r),$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\lambda^2-\frac{n^2}{r^2}+\frac{\rho}{\mu}f^2\right)w(r)=-\frac{\lambda}{1-2v}\Theta(r) \tag{13}$$

Let`s introduce the designation:

$$\nabla_*^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\lambda^2-\frac{n^2+1}{r^2}+\frac{\rho}{\mu}f^2. \tag{14}$$

Then the equations (13) will be over written as:

$$\nabla_*^2 u(r)-\frac{2n}{r^2}v(r)=-\frac{1}{1-2v}\frac{d\Theta(r)}{dr},$$

$$\nabla_*^2 v(r)-\frac{2n}{r^2}u(r)=\frac{n}{1-2v}\frac{1}{r}\Theta(r), \tag{15}$$

$$\left(\nabla_*^2+\frac{1}{r^2}\right)w(r)=-\frac{\lambda}{1-2v}\Theta(r).$$

The right parts of equations (15) contain functions or its derivative. Expression for a function, we find below from the Helmholtz equation (5) after the division of variables in it, according to the last formula (11). Thus, the right parts of equations (15) are known functions. Due to this circumstance these equations can be considered as heterogeneous equations concerning unknown functions such as *u(r), v(r), w(r).*

The general solution of the system of equations (15) in this case consists of the sum of the general decision of homogeneous system and any private solution heterogeneous. So, we find the general solution of the corresponding homogeneous system of equations:

$$\nabla_*^2 u(r)-\frac{2n}{r^2}v(r)=0;$$

$$\nabla_*^2 v(r) - \frac{2n}{r^2} u(r) = 0; \qquad (16)$$

$$\left(\nabla_*^2 + \frac{1}{r^2}\right) w(r) = 0.$$

We fold and subtract the left and right parts of the first two equations (16), then we get

$$\nabla_*^2 \left[u(r) - v(r)\right] + \frac{2n}{r^2}\left[u(r) - v(r)\right] = 0,$$

$$\nabla_*^2 \left[u(r) + v(r)\right] - \frac{2n}{r^2}\left[u(r) + v(r)\right] = 0. \qquad (17)$$

Let`s introduce the designation:

$$u(r) - v(r) = U(r), \quad u(r) + v(r) = V(r), \qquad (18)$$

So that

$$u(r) = \frac{1}{2}\left[U(r) + V(r)\right], \quad v(r) = \frac{1}{2}\left[V(r) - U(r)\right] \qquad (19)$$

Then the equations (17) will be overwritten as:

$$\nabla_*^2 U(r) + \frac{2n}{r^2} U(r) = 0, \quad \nabla_*^2 V(r) - \frac{2n}{r^2} V(r) = 0. \qquad (20)$$

We will solve these equations in the form of:

$$V(r) = 0 \qquad (21)$$

U(r)- Is the solution of the first of the equations (20), i.e.

$$\left(\nabla_*^2 + \frac{2n}{r^2}\right) U(r) = 0. \qquad (22)$$

Or considering (14)

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \lambda^2 - \frac{(n-1)^2}{r^2} + \frac{\rho}{\mu} f^2\right] U(r) = 0. \qquad (23)$$

Enter the designation:

$$\tilde{a} = \sqrt{\frac{\rho}{\mu} f^2 - \lambda^2}. \qquad (24)$$

Then the equation (23) will look like:

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+a^2-\frac{(n-1)^2}{r^2}\right]U(r)=0, \tag{25}$$

or

$$\left[r^2\frac{d^2}{dr^2}+r\frac{d}{dr}+a^2r^2-(n-1)^2\right]U(r)=0. \tag{26}$$

The equation (26) has a private solution in the form of (Loran et al., 2020:165302)

$$U(r)=Z_{(n-1)}(ra). \tag{27}$$

Where $Z_{(n-1)}(ra)$ is the cylindrical function

Considering (19), (27), we have:

$$u(r)=\frac{1}{2}U(r)=\frac{1}{2}Z_{(n-1)}(ra). \tag{28}$$

$$v(r)=-\frac{1}{2}U(r)=-\frac{1}{2}Z_{(n-1)}(ra).$$

Let's find the solution of the 3rd equation from (16):

$$\left(\nabla_n^2+\frac{1}{r^2}\right)w(r)=0. \tag{29}$$

or

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+a^2-\frac{n^2}{r^2}\right)w(r)=0. \tag{30}$$

$$\left(r^2\frac{d^2}{dr^2}+r\frac{d}{dr}+a^2r^2-n^2\right)w(r)=0. \tag{31}$$

The solution for:

w(r)=Zn(ra). (32)

Now let's solve the equation (15), which considering the last of the formulas (11), will look like:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\lambda^2-\frac{n^2}{r^2}+\frac{\rho}{2\mu}\frac{1-2\nu}{1-\nu}\right)\Theta(r)=0. \tag{33}$$

or

$$\left(r^2\frac{d^2}{dr^2}+r\frac{d}{dr}+b^2r^2-n^2\right)\Theta(r)=0, \quad (34)$$

Where

$$b=\sqrt{\frac{\rho}{2\mu}\frac{1-2\nu}{1-\nu}f^2-\lambda^2}. \quad (35)$$

$$\Theta(r)=Z_n(rb). \quad (36)$$

Consider the case = 1. Then you can write:

$$u(r)=-v(r)=\frac{1}{2}Z_0(ra)\cdot C_1, \quad w(r)=Z_1(ra)\cdot C_2, \quad (37)$$

$$\Theta(r)=Z_1(rb)\cdot C_3; \quad (38)$$

C_1, C_2, C_3– is arbitrary permanent integration. Here (Silva et al., 2021:116023):

$$Z_0(ra)=\sum_{s=0}^{\infty}\frac{(-1)^s}{(s!)^2}\left(\frac{ra}{2}\right)^{2s}=1-\frac{1}{(1!)^2}\left(\frac{ra}{2}\right)^2+\frac{1}{(2!)^2}\left(\frac{ra}{2}\right)^4-\frac{1}{(3!)^2}\left(\frac{ra}{2}\right)^6+$$
$$+\frac{1}{(4!)^2}\left(\frac{ra}{2}\right)^8-\frac{1}{(5!)^2}\left(\frac{ra}{2}\right)^{10}+\frac{1}{(6!)^2}\left(\frac{ra}{2}\right)^{12}-\frac{1}{(7!)^2}\left(\frac{ra}{2}\right)^{14}+\frac{1}{(8!)^2}\left(\frac{ra}{2}\right)^{16} \quad (39)$$

$$Z_1(ra)=\sum_{s=0}^{\infty}\frac{(-1)^s}{(1+s)!^2}\left(\frac{ra}{2}\right)^{1+2s}=\frac{1}{0!1!}\left(\frac{ra}{2}\right)-\frac{1}{1!2!}\left(\frac{ra}{2}\right)^3+\frac{1}{2!3!}\left(\frac{ra}{2}\right)^5-$$
$$-\frac{1}{3!4!}\left(\frac{ra}{2}\right)^7+\frac{1}{4!5!}\left(\frac{ra}{2}\right)^9-\frac{1}{5!6!}\left(\frac{ra}{2}\right)^{11}+\dots \quad (40)$$

In (40) *(ra)* should be replaced by $Z_1(rb)$.

Now we will find private solutions of heterogeneous equations (15) with the well-known right part.

Subtract and fold the left and right parts of the first two equations (15); Then we get:

$$\nabla_n^2[u_0(r)-v_0(r)]+\frac{2n}{r^2}[u_0(r)-v_0(r)]=-\frac{1}{1-2\nu}\left[\frac{n}{r}\Theta(r)+\frac{d\Theta(r)}{dr}\right], \quad (41)$$

$$\nabla_n^2[u_0(r)+v_0(r)]-\frac{2n}{r}[u_0(r)+v_0(r)]=\frac{1}{1-2\nu}\left[\frac{n}{r}\Theta(r)-\frac{d\Theta(r)}{dr}\right]. \quad (42)$$

Enter the designations:

$$u_0(r) - v_0(r) = U_0(r);\ u_0(r) + v_0(r) = V_0(r) \tag{43}$$

Here

$$u_0(r) = \frac{1}{2}\left[U_0(r) + V_0(r)\right],\ v_0(r) = \frac{1}{2}\left[V_0(r) - U_0(r)\right], \tag{44}$$

Equations (41) and (42) with consideration (43) will be written in the form of:

$$\left(\nabla_1^2 + \frac{2n}{r^2}\right)U_0(r) = -\frac{1}{1-2\nu}\left[\frac{n}{r}\Theta(r) + \frac{d\Theta(r)}{dr}\right], \tag{45}$$

$$\left(\nabla_1^2 - \frac{2n}{r^2}\right)V_0(r) = \frac{1}{1-2\nu}\left[\frac{n}{r}\Theta(r) - \frac{d\Theta(r)}{dr}\right], \tag{46}$$

Or, considering (14) in the form of:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \lambda^2 - \frac{(n-1)^2}{r^2} + \frac{\rho}{\mu}f^2\right]U_0(r) = -\frac{1}{1-2\nu}\left[\frac{n}{r}\Theta(r) + \frac{d\Theta(r)}{dr}\right], \tag{47}$$

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \lambda^2 - \frac{(n+1)^2}{r^2} + \frac{\rho}{\mu}f^2\right]V_0(r) = \frac{1}{1-2\nu}\left[\frac{n}{r}\Theta(r) - \frac{d\Theta(r)}{dr}\right], \tag{48}$$

Equations (47) and (48), considering the designation (24), take the form:

$$\left(r^2\frac{d^2}{dr^2} + r\frac{d}{dr} + a^2r^2\right)U_0(r) = \frac{1}{1-2\nu}\left[r\Theta(r) - r^2\frac{d\Theta(r)}{dr}\right], \tag{49}$$

$$\left(r^2\frac{d^2}{dr^2} + r\frac{d}{dr} + a^2r^2 - 4\right)V_0(r) = \frac{1}{1-2\nu}\left[r\Theta(r) - r^2\frac{d\Theta(r)}{dr}\right], \tag{50}$$

Let's write the equation (49) considering $\Theta(r) = Z_n(rb)C_3$, in the form of:

$$\left(r^2\frac{d^2}{dr^2} + r\frac{d}{dr} + a^2r^2\right)U_0(r) = \frac{C_3}{1-2\nu}\left(br^2 - \frac{r^4b^3}{4} + \frac{r^6b^5}{64} - \right)... \tag{51}$$

Suppose

$$U_0(r) = k_1r^2 + k_2r^4 + k_3r^6 + ... \tag{52}$$

Then

$$\frac{dU_0(r)}{dr} = 2k_1r + 4k_2r^3 + 6k_3r^5 + ... \tag{53}$$

$$\frac{d^2U_0(r)}{dr^2} = 2k_1r + 12k_2r^3 + 30k_3r^4 + ... \tag{54}$$

From the equation (51), considering (52)-(54) we get:

$$2k_1r^2 + 12k_2r^4 + 30k_3r^6 + ... + 2k_1r^2 + 4k_2r^4 + 6k_3r^6 + ... +$$

$$+a^2k_1r^4+a^2k_2r^6+a^2k_3r^8+\ldots=-\frac{C_3}{1-2v}\left(br^2-\frac{r^4b^3}{4}+\frac{r^6b^5}{64}-\ldots\right). \quad (55)$$

Equating coefficients at the same degrees zero get:

$$r^2: \quad 4k_1+\frac{bC_3}{1-2v}=0; \quad k_1=-\frac{C_3b}{4(1-2v)};$$

$$r^4: \quad 16k_2+a^2k_1-\frac{b^3C_3}{4(1-2v)}=0; \quad k_2=\frac{C_3b(a^2+b^2)}{64(1-2v)};$$

$$r^6: \quad 36k_3+a^2k_2+\frac{C_3b^5}{64(1-2v)}=0; \quad k_3=-\frac{C_3(b^5+a^2b^3+a^4b)}{36\cdot 64(1-2v)}. \quad (56)$$

Let's write the equation (50), considering in the form:

$$\left(r^2\frac{d^2}{dr^2}+r\frac{d}{dr}+a^2r^2-4\right)V_0(r)=\frac{1}{1-2v}\left[r\Theta(r)-r^2\frac{d\Theta(r)}{dr}\right] \quad (57)$$

Where

$$\Theta(r)=\frac{1}{1-2v}C_3\left[r^4\left(\frac{b}{2}\right)^3-\frac{1}{3}r^6\left(\frac{b}{2}\right)^5+\ldots\right]$$

Suppose

$$V_0(r)=m_1r^4+m_2r^6+\ldots \quad (58)$$

$$\frac{dV_0(r)}{dr}=4m_1r^3+6m_2r^5+\ldots,$$

Then

$$\frac{d^2V_0(r)}{dr^2}=12m_1r^2+30m_2r^4+\ldots \quad (59)$$

$$\frac{d^2V_0(r)}{dr^2}=12m_1r^2+30m_2r^4+\ldots \quad (60)$$

Substituting (58)-(60) in (57), we get:

$$12m_1r^4+30m_2r^6+\ldots+4m_1r^4+6m_2r^6+\ldots$$

$$-\frac{C_3}{1-2v}\left[r^4\left(\frac{b}{2}\right)^3-\frac{1}{3}r^6\left(\frac{b}{2}\right)^5+\ldots\right]=0 \quad (61)$$

Find

$$m_1 = \frac{C_1 b^3}{96(1-2v)}; \quad m_2 = -\frac{C_1 b^3 (a^2+b^2)}{32 \cdot 96(1-2v)} \ldots; \tag{62}$$

Let’s write the last of the equations (15):

$$\left(\nabla_*^2 + \frac{1}{r^2}\right) w(r) = -\frac{\lambda}{1-2v}\Theta(r); \tag{63}$$

Or, considering (14), (24) – In the form of:

$$\left(r^2 \frac{d_2}{dr^2} + r\frac{d}{dr} + a^2 r^2 - n^2\right) w(r) = -\frac{\lambda r^2}{1-2v}\Theta(r). \tag{64}$$

with $n=1$ we get:

$$\left(r^2 \frac{d_2}{dr^2} + r\frac{d}{dr} + a^2 r^2 - 1\right) w(r) = -\frac{\lambda r^2}{1-2v}\Theta(r). \tag{65}$$

Considering (38), supposing a private solution to the equation ((65) in the form of:

$$w(r) = n_1 r^3 + n_2 r^5 + n_3 r^7 + \ldots \tag{66}$$

$$\frac{dw(r)}{dr} = 3n_1 r^2 + 5n_2 r^4 + 7n_3 r^6 + \ldots, \tag{67}$$

$$\frac{d^2 w(r)}{dr^2} = 6n_1 r + 20n_2 r^3 + 42n_3 r^5 + \ldots, \tag{68}$$

Get out of the equation (65):

$$6n_1 r^3 + 20n_2 r^5 + 42n_3 r^7 + \ldots + 3n_1 r^3 + 5n_2 r^5 + 7n_3 r^7 + a^2 n_1 r^5 + a^2 n_2 r^7 +$$

$$+ a^2 n_3 r^9 - n_1 r^3 - n_2 r^5 - n_3 r^7 - \ldots = -\frac{\lambda C_1}{1-2v}\left(r^3 \frac{b}{2} - \frac{1}{2} r^5 \frac{b^3}{2^3} + \frac{1}{12} r^7 \frac{b^5}{2^5} - \ldots\right) \tag{69}$$

Here we find:

$$n_1 = -\frac{\lambda C_3 b}{16(1-2v)}; \qquad n_2 = \frac{C_3 \lambda b(a^2+b^2)}{16 \cdot 24(1-2v)}; \; n_3 = -\frac{\lambda C_3 b(a^4+a^2b^2+b^4)}{16 \cdot 24 \cdot 48(1-2v)} \tag{70}$$

Result and discussion

So, let’s write the solution of dynamic equations of free oscillations of the system. The sought displacements, considering (37), (38), (52)-(56), (58)-(62), (66)-(70), shall be of the following type:

$$u(r) = \frac{1}{2} Z_0(ra) C_1 + \frac{1}{2} C_3 \left[U_0(r) + V_0(r)\right];$$

$$v(r) = -\frac{1}{2}Z_0(ra)C_1 + \frac{1}{2}C_3\left[V_0(r) - U_0(r)\right]; \qquad (71)$$

$$w(r) = C_2 Z_0(ra) + C_3 w(r).$$

Let's write down the boundary conditions of free oscillations of the considered system for a continuous cylindrical body at $r = r_0$, consisting in equal zero of stresses on lateral surface of it:

$$\sigma_{rr}\Big|_{r=r_0} = 2\mu\left[\frac{\nu}{1-2\nu}\Theta(r)\sin\lambda z\cos n\varphi + \frac{du(r)}{dr}\sin\lambda z\cos n\varphi\right]_{r=r_0} = 0,$$

or

$$\left[\frac{\nu}{1-2\nu}\Theta(r) + \frac{du(r)}{dr}\right]_{r=r_0} = 0; \qquad (72)$$

or

$$\sigma_{r\varphi}\Big|_{r=r_0} = \mu\left[\frac{dv(r)}{dr}\sin\lambda z\sin n\varphi - \frac{n}{r}u(r)\sin\lambda z\sin n\varphi - \frac{v(r)}{r}\sin\lambda z\sin n\varphi\right]_{r=r_0} = 0,$$

or

$$\left[\frac{dv(r)}{dr} - \frac{n}{r}u(r) - \frac{v(r)}{r}\right]_{r=r_0} = 0; \qquad (73)$$

or

$$\sigma_{rz} = \mu\left[\lambda u(r)\cos n\varphi - \frac{dw(r)}{dr}\cos\lambda z\cos n\varphi\right]_{r=r_0} = 0,$$

or

$$\left[\lambda u(r) - \frac{dw(r)}{dr}\right]_{r=r_0} = 0, \qquad (74)$$

where $r = r_0$, is is the outer radius of the solid cylindrical body.

Substituting (71) in (72)–(74), we shall receive the system of three homogeneous algebraic equations concerning three unknown arbitrary constant, non-trivial (nonzero) decision of which takes place at equality of zero of its characteristic determinant. Periods of free oscillations of the system can be found from the characteristic equation.

Considering the inertial terms, an internal boundary value problem is solved for excitations in a cylindrical coordinate system. The Tedone equations of motion are solved for the compressible material of cylindrical bodies. Solutions of homogeneous differential equations in Bessel functions are obtained by preliminary separation of variables of inhomogeneous equations determined by the method of indefinite coefficients.

The Tedone equation of motion is solved in the case of compression of cylindrical material bodies. In this case, the volume expansion obeys the Helmholtz equations. Solutions of homogeneous differential equations in Bessel functions are created in the work, in which the variables of non-homogeneous equations are separated in advance. These homogeneous differential equations are found by the method of indefinite coefficients.

In addition, the internal boundary value problem of excitation is solved. The Tedone equation of motion is solved for the compressed material of some cylindrical bodies. The solution of homogeneous differential equations in Bessel functions is obtained by preliminary separation of variables. It is noted that the solution of inhomogeneous differential equations is obtained by the method of indefinite coefficients.

Conclusion

Thus, in this paper, the internal boundary value problem for perturbations in a cylindrical coordinate system is solved, considering the inertial terms. The equation in displacements, presented in the Tedone form, is solved for the case of compressible material of cylindrical bodies. The volumetric expansion obeys the Helmholtz equation. Solutions of homogeneous differential equations are obtained in Bessel functions with preliminary separation of variables of inhomogeneous equations, which are found by the method of indefinite coefficients.

REFERENCES

Dossumbekov K.R., Ispulov N.A., Kurmanov A.A. and Zhumabekov A.Zh., 2021 — *Dossumbekov K.R., Ispulov N.A., Kurmanov A.A. and Zhumabekov A.Zh.* Propagation of electromagnetic waves in cholesteric liquid crystals, Russian Physics Journal, Vol. 64. №. 8. DOI: 10.1007/s11182-021/02470-8.

Ispulov N.A. Qadir A., Shah M.A., Seythanova A.K., Kissikov T.G., Arinov E., 2016 — *Ispulov N.A. Qadir A., Shah M.A., Seythanova A.K., Kissikov T.G., Arinov E.* Reflection of thermoelastic wave on the interface of isotropic half-space and tetragonal syngony anisotropic medium of classes 4, 4/m with thermomechanical effect, Chinese Physics B, Number of article: 038102. DOI: 10.1088/1674-1056/25/3/038102.

Ispulov N.A., Qadir A., Zhukenov M.K., Arinov E., 2017 — *Ispulov N.A. Qadir A., Zhukenov M.K., Arinov E.* The Propagation of Thermoelastic Waves in Anisotropic Media of Orthorhombic, Hexagonal, and Tetragonal Syngonies, Advances in Mathematical Physics, Number of article: 4898467. DOI: 10.1155/2017/4898467.

Ispulov N.A., Abdul Qadir, Zhukenov M.K., Dossanov T.S., Kissikov T.G., 2017 — *Ispulov N.A., Abdul Qadir, Zhukenov M.K., Dossanov T.S., Kissikov T.G.* The analytical form of the dispersion equation of elastic waves in periodically inhomogeneous medium of different classes of crystals. Advances in Mathematical Physics, 24. 5236898. DOI:10.1155/2017/5236898.

Ispulov N.A., Zhumabekov A.Zh., Abdul Qadir, Kurmanov A.A., Sarymova Sh.N., Dossumbekov K.R., Arinov E., 2022 — *Ispulov N.A., Zhumabekov A.Zh., Abdul Qadir, Kurmanov A.A., Sarymova Sh.N., Dossumbekov K.R., Arinov E.* The Propagation of Thermoelastic Waves in Different Anisotropic Media Using Matricant Method. Advances in Mathematical Physics. 5787899. DOI:10.1155/2022/5787899.

Ispulov N.A., Zhumabekov A.Zh., Dossumbekov K.R., Bektazinova A.K., 2022 — *Ispulov N.A., Zhumabekov A.Zh., Dossumbekov K.R., Bektazinova A.K.* On matrices of coefficients of electromagnetic and elastic waves propagating in anisotropic media. Bulgarian Chemical Communications, 54, Special Issue B1. 53–57. DOI: 10.34049/bcc. 54.

Ispulov N.A., Abdul Qadir, Zhumabekov A.Zh., Kurmanov A.A., Dosumbekov K.R., 2022 —

Ispulov N.A., Abdul Qadir, Zhumabekov A.Zh., Kurmanov A.A., Dosumbekov K.R. On nonclassical boundary conditions for the contact of thin interlayers with different physical and mechanical properties on wave propagation in anisotropic media. Bulletin of the Karaganda University. Physics Series. № 3. DOI: 10.31489/2022PH3/68–79.

Kurmanov A.A., Ispulov N.A., Abdul Qadir, Zhumabekov A.Zh., Sarymova Sh.N., Dossumbekov K.R., 2021 — *Kurmanov A.A., Ispulov N.A., Abdul Qadir, ZhumabekovA.Zh., Sarymova Sh.N., Dossumbekov K.R.* Propagation of electromagnetic waves in stationary anisotropic media, Physica Scripta, 96. 085505. DOI:10.1088/1402-4896/abfe87.

Li S., Xu P., Xu Y., 2021 — *Li S., Xu P., Xu Y.* Local photonic density of states in hyperbolic metasurfaces. Journal of Optics, 23. 115101. DOI:10.1088/2040-8986/ac27bc.

Mussa E.U., Ispulov N.A., Dossumbekov K.R., Zhumabekov A.Zh., 2022 — *Mussa E.U., Ispulov N.A., Dossumbekov K.R., Zhumabekov A.Zh.* On the propagation of thermoelastic waves in anisotropic media (inhomogeneity along the Y-axis). Bulletin ToU. Physics, mathematics and computer science. 1. 58–70. DOI: 10.48081/NBDT3831.

Loran F., Mostafazadeh A., 2020 — *Loran F., Mostafazadeh A.* Transfer-matrix formulation of the scattering of electromagnetic waves and broadband invisibility in three dimensions. Journal of Physics A-Mathematical and Theoretical, 53. 165302. DOI:10.1088/1751-8121/ab7669.

Lurie A.I., 2005 — *Lurie A.I.* Theory of Elasticity, Springer.

Nowacki W., 1986 — *Nowacki W.* Thermoelasticity. 2nd edition. Pergamon Press, Oxford.

Silva P.D.S., Lisboa-Santos L., Ferreira M.M.Jr., Schreck M., 2021 — *Silva P.D.S., Lisboa-Santos L., Ferreira M.M.Jr., Schreck M.* Effects of CPT-odd terms of dimensions three and five on electromagnetic propagation in continuous matter. Physical Review D. 104. 116023. DOI:10.1103/PhysRevD.104.116023.

Sharma M.D., 2004 — *Sharma M.D.* Three-dimensional wave propagation in a general anisotropic poroelastic medium: phase velocity, group velocity and polarization. Geophysical Journal International, 156. 329–344. DOI:10.1111/j.1365-246X.2003.02141.x.

Tleukenov S.K., 2004 — Tleukenov S.K. Matricant Method, PSU press, Pavlodar.

Tleukenov S.K., Zhukenov M.K., Ispulov N.A., 2019 — *Tleukenov S.K., Zhukenov M.K., Ispulov N.A.* Propagation of electromagnetic waves in anisotropic magnetoelectric medium Bulletin of the Karaganda University. Physics Series. № 2. DOI: 10.31489/2019Ph2/29-34.

CHEMISTRY